Axioms

I. Propositional axioms

* All tautologies[2]

II. Quantifier axioms[3]

(1) Universal specification (US): $\forall x\, P(x) \rightarrow P(t)$, where t is any term or expression of the same sort as the variable x and no free variable of t becomes bound when P(t) is formed.[4]

(2) De Morgan's law for quantifiers: $\sim \forall x\, P(x) \leftrightarrow \exists x \sim P(x)$

*(3) De Morgan's law for quantifiers: $\sim \exists x\, P(x) \leftrightarrow \forall x \sim P(x)$

*(4) Existential Generalization (EG): $P(t) \rightarrow \exists x\, P(x)$, with the same restrictions on t as in universal specification.

III. Equality axioms

(1) Reflexivity: $x = x$

*(2) Symmetry: $x = y \rightarrow y = x$

*(3) Transitivity: $(x = y \wedge y = z) \rightarrow x = z$

(4) Substitution of equals: $x = y \rightarrow [P(x) \leftrightarrow P(y)]$, where $P(y)$ results from replacing *some or all* of the occurrences of x in $P(x)$ with y and neither x nor y is quantified in $P(x)$.[5]

IV. Set axioms

In these axioms, the variables A, B, C, and D denote sets but w, x, y, and z can be any sort of objects, not necessarily real numbers. This group of axioms is included here primarily for completeness; most of them are not discussed in the text.

(1) Extensionality: $A = B \leftrightarrow \forall x\, (x \in A \leftrightarrow x \in B)$

(2) Pairing: $\forall x, y\, \exists A\, \forall z\, [z \in A \leftrightarrow (z = x \vee z = y)]$. (Less formally, this says: for every x and y, the set $\{x, y\}$ exists.)

(3) There is a set $\mathbb{R}$ of all real numbers.[6]

*(4) For every x and y, the ordered pair (x, y) exists.

*(5) $(w, x) = (y, z)$ iff $w = y$ and $x = z$.

(6) Power set axiom: $\forall A\ \exists B\ \forall C\ (C \in B \leftrightarrow C \subseteq A)$. (Less formally, this says: for every set A, $\wp(A)$ exists.)

(7) Union axiom: $\forall \mathscr{A}\ \exists B\ \forall x\ [x \in B \leftrightarrow \exists C\ (C \in \mathscr{A} \land x \in C)]$. (Less formally, this says: for every set of sets $\mathscr{A}$, the union of all the sets in $\mathscr{A}$ ($\bigcup \mathscr{A}$) exists.)

*(8) Separation axiom: For every proposition $P(x)$ and every set A, the set $\{x \in A \mid P(x)\}$ exists.

(9) Replacement axiom: For every proposition $P(x, y)$ and every set A,

$$[\forall x \in A\ \exists! y\ P(x, y)] \rightarrow \exists B\ \forall y\ [y \in B \leftrightarrow \exists x \in A\ P(x, y)]$$

(Less formally, this says: if $P(x, y)$ defines a function whose domain is the set A, then its range is also a set.)

(10) Foundation axiom: $\forall A\ [A \neq \varnothing \rightarrow \exists B \in A\ (B \cap A = \varnothing)]$

(11) Axiom of choice (AC): For every collection $\mathscr{A}$ of nonempty sets, there is a function f such that, for every B in $\mathscr{A}$, $f(B) \in B$.

V. Real number axioms

In these axioms, x, y, and z denote real numbers. Axioms 1–12 of this group are called the **field axioms**, while axioms 1–17 are called the **ordered field axioms**.

(1) Additive closure: $\forall x, y\ \exists z\ (x + y = z)$

(2) Multiplicative closure: $\forall x, y\ \exists z\ (x \cdot y = z)$

(3) Additive associativity: $x + (y + z) = (x + y) + z$

(4) Multiplicative associativity: $x \cdot (y \cdot z) = (x \cdot y) \cdot z$

(5) Additive commutativity: $x + y = y + x$

(6) Multiplicative commutativity: $x \cdot y = y \cdot x$

(7) Distributivity: $x \cdot (y + z) = (x \cdot y) + (x \cdot z)$ and $(y + z) \cdot x = (y \cdot x) + (z \cdot x)$[7]

(8) Additive identity: There is a number, denoted 0, such that for all x, $x + 0 = x$.[8]

(9) Multiplicative identity: There is a number, denoted 1, such that for all x, $x \cdot 1 = 1 \cdot x = x$.[7,8]

(10) Additive inverses: For every x there is a number, denoted $-x$, such that $x + (-x) = 0$.[8]

(11) Multiplicative inverses: For every nonzero x there is a number, denoted x^{-1}, such that $x \cdot x^{-1} = x^{-1} \cdot x = 1$.[7,8]

(12) $0 \neq 1$

(13) Irreflexivity of $<$: $\sim(x < x)$

(14) Transitivity of $<$: If $x < y$ and $y < z$, then $x < z$

(15) Trichotomy: Either $x < y$, $y < x$, or $x = y$

(16) If $x < y$, then $x + z < y + z$

(17) If $x < y$ and $0 < z$, then $x \cdot z < y \cdot z$ and $z \cdot x < z \cdot y$[7]

(18) Completeness: If a nonempty set of real numbers has an upper bound, then it has a *least* upper bound.

VI. Natural number axioms

In these axioms, m and n denote natural numbers and A denotes a set.

(1) $1 \in \mathbb{N}$[9]

(2) If $m \in \mathbb{N}$, then $m + 1 \in \mathbb{N}$.

(3) Mathematical induction (set form): $[1 \in A \wedge \forall n\,(n \in A \rightarrow n + 1 \in A)] \rightarrow \mathbb{N} \subseteq A$

*(3′) Mathematical induction (statement form):

$$[\mathrm{P}(1) \wedge \forall n\,(\mathrm{P}(n) \rightarrow \mathrm{P}(n+1))] \rightarrow \forall n\,\mathrm{P}(n)$$ [10]

Footnotes

(1) Various other derived rules of inference, which may be used as if they were part of the axiom system, are given in Section 4.2.

(2) For a list of many useful tautologies, see Appendix 3.