Example 1. Let a and b be non-zero real numbers and compute

$$\int e^{ax}\cos bx\ dx.$$

Solution: Let $u = \cos bx$, $dv = e^{ax}\ dx$, and integrate by parts.

$$\int e^{ax}\cos bx\ dx = \frac{1}{a}e^{ax}\cos bx + \frac{b}{a}\int e^{ax}\sin bx\ dx. \tag{1}$$

To compute $\int e^{ax}\sin bx\ dx$, we again let $dv = e^{ax}\ dx$, but this time pick $u = \sin bx$. Then

$$\int e^{ax}\sin bx\ dx = \frac{1}{a}e^{ax}\sin bx - \frac{b}{a}\int e^{ax}\cos bx\ dx. \tag{2}$$

Plugging equation (2) into equation (1), we see that

$$\begin{aligned}
\int e^{ax}\cos bx\ dx &= \frac{1}{a}e^{ax}\cos bx + \frac{b}{a}\left(\frac{1}{a}e^{ax}\sin bx - \frac{b}{a}\int e^{ax}\cos bx\ dx\right)\\
\int e^{ax}\cos bx\ dx &= \frac{1}{a}e^{ax}\cos bx + \frac{b}{a^2}e^{ax}\sin bx - \frac{b^2}{a^2}\int e^{ax}\cos bx\ dx\\
\left(1+\frac{b^2}{a^2}\right)\int e^{ax}\cos bx\ dx &= \frac{1}{a}e^{ax}\cos bx + \frac{b}{a^2}e^{ax}\sin bx\\
\left(\frac{a^2+b^2}{a^2}\right)\int e^{ax}\cos bx\ dx &= \frac{1}{a^2}e^{ax}\left(a\cos bx + b\sin bx\right).
\end{aligned}$$

Finally, cancel the a^2 in the denominator of each side and divide by a^2+b^2 (which is non-zero since a, $b \neq 0$) to arrive at

$$\boxed{\int e^{ax}\cos bx\ dx = \frac{1}{a^2+b^2}e^{ax}\left(a\cos bx + b\sin bx\right) + C}$$

Remark: The most interesting case is the case where a, b are both non-zero, as in the statement of the problem. If $a = 0$ but $b \neq 0$, then $e^{ax} = 1$, so the integral reduces to the much simpler $\int \cos bx\ dx = \frac{1}{b}\sin bx + C$. If $b = 0$ but $a \neq 0$, then $\cos bx = 1$, and the integral is $\int e^{ax}\ dx = \frac{1}{a}e^{ax} + C$. Finally, if $a = b = 0$, then the integral is $\int 1\ dx = x + C$.

Example 2. Let a and b be as in example 1. Compute

$$\int e^{ax}\sin bx\ dx.$$

SOLUTION: Instead of plugging (2) into (1) as in example 1, plug (1) into (2) to get

$$\int e^{ax}\sin bx\ dx = \frac{1}{a}e^{ax}\sin bx - \frac{b}{a}\left(\frac{1}{a}e^{ax}\cos bx + \frac{b}{a}\int e^{ax}\sin bx\ dx\right)$$
$$\int e^{ax}\sin bx\ dx = \frac{1}{a}e^{ax}\sin bx - \frac{b}{a^2}e^{ax}\cos bx - \frac{b^2}{a^2}\int e^{ax}\sin bx\ dx.$$

After performing algebraic manipulations similar to example 1, we arrive at

$$\boxed{\int e^{ax}\sin bx\ dx = \frac{1}{a^2+b^2}e^{ax}\left(a\sin bx - b\cos bx\right) + C}$$

So what? Why suffer through the algebra above to arrive at such a messy answer? Well, now if you have to integrate something like $\int e^{-x}\cos 3x\ dx$ on an exam and you remember the formula, you can just plug in $a=-1$ and $b=3$ into the answer from example 1 to get

$$\int e^{-x}\cos 3x\ dx = \frac{1}{10}e^{-x}\left(-\cos 3x + 3\sin 3x\right) + C.$$

Remember: there's no work required for a multiple choice exam, so you can skip right to the answer if you have a good enough memory!