CprE 528, Spring 2014 Probabilistic Methods in Computer Engineering

Homework 1

Due: Thu, 30 Jan

Policy

This is a *collaborative* homework. You are allowed to discuss the problems with other students. But, the write up should be written by yourself without help from others.

Problem 1.1

Let a, b be two numbers chosen independently at random (with replacement) from the set $\{1, 2, \ldots, 10\}$. Among the following three events, which pairs of events are independent? Explain. Note that you need to check all three pairs of events.

$A = \{a \leq 5\}, B = \{a \neq b\}, C = \{a + b < 6\}$

Problem 1.2

Suppose you are given a coin for which the probability of heads (say, p) is unknown. Show how you can use this to simulate unbiased coin flips (i.e. a coin which comes out heads or tails with equal probability). Assume that $0 < p < 1$.

By "simulate", we mean the following. Assume that you have a procedure A which behaves like the coin given to you; its result is 1 or 0 with probability p and $1 - p$ respectively, where p is unknown to you. Use procedure A to write another procedure, B which behaves like an unbiased coin. The result of procedure B should be 0 or 1 with equal probability.

Hint: Consider two consecutive flips of the biased coin.

Problem 1.3

Prove the following union bound using Mathematical Induction. For any set of k events $E_1, E_2, \ldots, E_k$, prove:

$$\Pr\left(\bigcup_{i=1}^{k} E_i\right) \leq \sum_{i=1}^{k} \Pr(E_i)$$

Problem 1.4

- You are given a procedure `RandomBit()` which returns a random bit (0 or 1) with equal probability. Assume that the results of different calls to `RandomBit()` are independent of each other. Using `RandomBit()` as a subroutine, write a procedure `RandomInt(a,b)`, which takes as input two positive integers $a, b, a \leq b$, and returns an integer chosen uniformly and independently at random from the set $\{a, a+1, \ldots, b\}$. How many calls does `RandomInt(a,b)` need to make to `RandomBit()`?
- Using `RandomInt(a,b)` as a subroutine, write a procedure `Permute(n)` which returns a random permutation of the set $\{1, 2, 3, \ldots, n\}$. How many calls does `Permute(n)` need to make to `RandomInt(a,b)`?

Problem 1.5

Generalizing the notion of a cut-set, we define a 3-way cut-set in a graph as a set of edges whose removal breaks the graph into 3 or more connected components. Explain how the randomized min-cut algorithm (which we discussed in class) can be modified to find a minimum size 3-way cut-set in a graph, and bound the probability that the algorithm succeeds in one iteration.