CS 357 / MATH 357
Spring 2014

Numerical Methods – Exam 2 practice problems
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Note: these problems are from "Numerical Mathematics and Computing" seventh edition.

Problem 1 [0pt] How many binary digits of precision are gained in each step of the bisection method? How many steps are required for each decimal digit of precision?

> If the original interval is of width h, then after k steip, we have reduced the interval containing the root to width $h2^{-k}$. From then on, we add one bit at each step. About three steps are needed for each decimal digit

Problem 2 [0pt] Write Newton's method in simplified form for determining the reciprocal of the square root of a positive number. Perform two iterations to approximate $1/\pm\sqrt{5}$, starting with $x_0 = 1$ and $x_0 = -1$.

> $x_{n+1} = \frac{1}{2}\left[x_n + \frac{1}{(Rx_n)}\right]$

Problem 3 [0pt] Establish Newton's iterative scheme in simplfied form, not involving the reciprocal of x, for the function $f(x) = xR - x^{-1}$. Carry out three steps of this procedure using $R = 4$ and $x_0 = -1$.

> $x_{n+1} = \frac{2x_n}{x_n^2 R+1}$; -0.49985

Problem 4 [0pt] For what starting values will Newton's method converge if the function f is $f(x) = x^2/(1 + x^2)$

> $|x_0| < \sqrt{3}$

Problem 5 [0pt] The reciprocal of a number R can be computed without division by the iterative formula $x_{n+1} = x_n(2 - x_nR)$. Establish this relation by applying Newton's method to some $f(x)$. Beginning with $x_0 = 0.2$, compute the reciprocal of 4 correct to six decimal digits or more by this rule. Tabulate the error at each step and observe the quadratic convergence.

Problem 6 [0pt] Show that Newton's method applied to $x^m - R$ and to $1-(R/x^m)$ for determining $\sqrt[m]{R}$ results in two similar yet different iterative formulas. Here $R > 0, m \geq 2$.

> $x_{n+1} = [(m-1)x_n^m + R] / \left(mx_n^{m-1}\right)$; $x_{n+1} = x_n [(m+1)R - x_n^m] / (mR)$

Problem 7 [0pt] Perform one iteration of the multidimensional Newton's method for the following system of nonlinear equations.

$$f(\boldsymbol{x}) = \begin{bmatrix} x^2 + y \\ y^2 - 4 \end{bmatrix}$$

with

$$\boldsymbol{x_0} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

What is $\boldsymbol{x_1}$?

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix} = \begin{bmatrix} 2x & 1 \\ 0 & 2y \end{bmatrix}$$

$$\boldsymbol{x_{n+1}} - \boldsymbol{x_n} = -J^{-1} f(\boldsymbol{x_n})$$

solve

$$\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \delta x \\ \delta y \end{bmatrix} = f(\boldsymbol{x_0}) = \begin{bmatrix} 2 \\ -3 \end{bmatrix} \to \begin{bmatrix} \delta x \\ \delta y \end{bmatrix} = \begin{bmatrix} 7/4 \\ -3/2 \end{bmatrix}$$

$$\boldsymbol{x_1} = \boldsymbol{x_0} - \begin{bmatrix} \delta x \\ \delta y \end{bmatrix} = \begin{bmatrix} -3/4 \\ 5/2 \end{bmatrix}$$

Problem 8 [0pt] The Taylor series for a function f looks like this:

$$f(x+h) = f(x) + hf^{'}(x) + \frac{h^2}{2} f^{''}(x) + \frac{h^3}{6} f^{'''}(x) + \cdots$$

Suppose that $f(x)$, $f^{'}(x)$, and $f^{''}(x)$ are easily computed. Derive an algorithm like Newton's method that uses three terms in the Taylor series. The algorithm shoucl take as input an approximation to the root and produce as output a better approximation to the root. Show that the method is cubically convergent.

$$x_{n+1} = x_n - \frac{f^{'}(x_n)}{f^{''}(x_n)} + \frac{\sqrt{[f^{'}(x_n)]^2 - 2f(x_n)f^{''}(x_n)}}{f^{''}(x_n)}$$

Problem 9 [0pt] What is the appropriate formula for finding square roots using the secant method?

$$x_{n+1} = x_n - \frac{x_n^2 - R}{x_n + x_{n-1}}$$

Problem 10 [0pt] A method of finding a zero of a given function f proceeds as follows. Two initial approximations x_0 and x_1 to the zero are chosen, the value of x_0 is fixed, and successive iterations are given by

$$x_{n+1} = x_n - \left(\frac{x_n - x_0}{f(x_n) - f(x_0)}\right) f(x_n).$$

This process will converge to a zero of f uncer certain conditions. Show that the rate of convergence to a simple zero is linear under some conditions.

$$e_{n+1} = \left[1 - \left(\frac{x_n - x_0}{f(x_n) - f(x_0)}\right) f'(\xi_n)\right] e_n$$

Problem 11 [0pt] Use the Lagrange interpolation process to obtain a polynomial of least degree that assumes these values:

x	0	2	3	4
y	7	11	28	63

$$p_3(x) = 7 - 2x + x^3$$

Problem 12 [0pt] For the four interpolation nodes -1, 1, 3, 4, what are the ℓ_k functions required in the Lagrange interpolation procedure? Find ℓ_2.

$$\ell_2(x) = -(x-4)(x^2-1)/8$$

Problem 13 [0pt] Complete the following divided difference table and use it to obtain polynomials of degree 3 that interpolate the function values indicated:

x	$f[]$	$f[,]$	$f[,,]$	$f[,,,]$
-1	2			
1	-4		2	
3	6			
		2		
5	10			

$$p_3(x) = 2 + (x+1)\left(-3 + (x-1)\left(2 - (x-3)\frac{11}{24}\right)\right)$$

Problem 14 [0pt] Given the data

x	0	1	2	4	6
y	1	9	23	93	259

construct the divided difference table.

x	$f[]$	$f[,]$	$f[,,]$	$f[,,,]$	$f[,,,,]$
0	1				
		8			
1	9		3		
		14		1	
2	23		7		0
		35		1	
4	93		12		
		83			
6	259				

Problem 15 [0pt] Using Newton's interpolation polynomial, find an approximation to $f(4.2)$ using the data from the previous example.

$$f(4.2) = 104.488$$

Problem 16 [0pt] There exists a unique polynomial p(x) of degree 2 or less such that $p(0) = 0$ $p(1) = 1$ $p^{'}(\alpha) = 2$ for any value of α between 0 and 1 (inclusive) except one value of α, say α_0. Determine α_0, and give this polynomial for $\alpha \neq \alpha_0$.

$$\alpha_0 = \frac{1}{2}$$

Problem 17 [0pt] Find the polynomial of least degree that interpolates these values:

x	1.73	1.82	2.61	5.22	8.26
y	0	0	7.8	0	0

Hint: Rearrange the table so that the nonzero value of y is the last entry.

$$p(x) = 0.76(x - 1.73)(x - 1.82)(x - 5.22)(x - 8.26)$$

Problem 18 [0pt] Find the polynomial $p(x)$ of degree at most 3 such that $p(0) = 1,\ p(1) = 0,\ p^{'}(0) = 0,$ and $p^{'}(-1) = -1$.

$$p(x) = -\frac{3}{5}x^3 - \frac{2}{5}x^2 + 1$$

Problem 19 **[0pt]** An interpolating polynomial of degree 20 is to be used to approxomate e^{-x} on the interval $[0, 2]$. How accurate will it be? (use 21 univorm nodes, including the endpoints of the interval)

The error E will satisfy $E \leq 4.105 \times 10^{-14}$

Problem 20 **[0pt]** Let the function $f(x) = \ln x$ be approximated by an interpolation polynomial of degree 9 with 10 nodes uniformly distributed in the interval $[1, 2]$. What bound can be placed on the error?

2.6×10^{-6}

Problem 21 **[0pt]** Determine the error term for the formula:

$$f^{'}(x) \approx \frac{1}{4h}[f(x+3h) - f(x-h)]$$

$-hf^{''}(\xi)$

Problem 22 **[0pt]** Using Taylor series, establish the error term for the formula:

$$f^{'}(0) \approx \frac{1}{2h}[f(2h) - f(0)]$$

Error term $= -hf^{''}(\xi)$ for $\xi \in (0, 2h)$

Problem 23 **[0pt]** Criticize the following analysis. By Taylor's formula, we have

$$f(x+h) - f(x) = hf^{'}(x) + \frac{h^2}{2}f^{''}(x) + \frac{h^3}{6}f^{'''}(\xi)$$

$$f(x-h) - f(x) = -hf^{'}(x) + \frac{h^2}{2}f^{''}(x) - \frac{h^3}{6}f^{'''}(\xi)$$

so by adding we obtain an exact expression for $f''(x)$:

$$f(x+h) + f(x-h) - 2f(x) = h^2 f''(x)$$

The point ξ for the first Taylor series is such that $\xi \in (x, x+h)$, while the second is $\xi \in (x-h, x)$. They are not the same.

Problem 24 [0pt] Derive the following rules for these estimating derivatives and establish their error terms.

$$f'''(x) \approx \frac{1}{2h^3}[f(x+2h) - 2f(x+h) + 2f(x-h) - f(x-2h)]$$

$$f^{(4)}(x) \approx \frac{1}{h^4}[f(x+2h) - 4f(x+h) + 6f(x) - 4f(x-h) + f(x-2h)]$$

Hint: Consider the Taylor series for $D(h) \equiv f(x+h) - f(x-h)$ and $S(h) \equiv f(x+h) + f(x-h)$

The error terms are:

$$-\frac{h^2}{4} f^{(5)}(\xi)$$

and

$$-\frac{h^2}{6} f^{(6)}(\xi)$$

respectively.

Problem 25 [0pt] What formula results from using the composite trapezoid rule on $f(x) = x^2$ with interval $[0, 1]$ and n+1 equally spaced points? Simplify your result by using the fact that $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{1}{6}n(2n+1)(n+1)$. Show that as $n \to \infty$, the trapezoid estimate converges to the correct value, $\frac{1}{3}$

$$T = \frac{1}{n^3}\left[\frac{1}{6}(n-1)(2n-1)n\right] + \frac{1}{2n}$$

Problem 26 [0pt] Consider $\int_0^1 \frac{1}{(x^2+2)} dx$. What is the result of using the composite trapezoid rule with 0, $\frac{1}{2}$ and 1 as partition points?

$$T(f\colon P) \approx 0.43056$$

Problem 27 [0pt] Compute $\int_0^1 (1+x^2)^{-1} dx$ by the basic simpson's using the three partition points $x = 0, 0.5$, and 1. Compare with the true solution.

$\frac{\pi}{4}$

Problem 28 [0pt] Find an approximate value of $\int_1^2 x^{-1} dx$ using composite simpson's rule with h = 0.25. Give a bound on the error.

$$\int_1^2 \frac{dx}{x} = 0.6933$$

Error bound is 5.2×10^{-4}

Problem 29 [0pt] A Gaussian quadrature rule for the interval $[-1,1]$ can be used on the interval $[a,b]$ by applying a suitable linear transformation. Approximate

$$\int_0^2 e^{-x^2}\,dx$$

using the transformed rule with weights $w_0 = 1 = w_1$ and nodes $x_0 = -\frac{1}{\sqrt{3}}$ $x_1 = \frac{1}{\sqrt{3}}$

$$\approx 0.91949$$

Problem 30 [0pt] Construct a rule of the form

$$\int_{-1}^{1} f(x)dx \approx \alpha f(-\frac{1}{2}) + \beta f(0) + \gamma f(\frac{1}{2})$$

that is exact for all polynomials of degree ≤ 2; that is determine values for α, β, and γ.

Hint: Make the result exact for 1, x, and x^2 and find a solution of th resulting equations. If it is exact for these polynomials it is exact for polynomials of degree ≤ 2

$$\alpha = \gamma = \frac{4}{3},\ \beta = -\frac{2}{3}$$

Problem 31 [0pt] Derive the Gaussian quadrature rule of the form

$$\int_{-1}^{1} f(x)x^2dx \approx af(-\alpha) + bf(0) + cf(\alpha)$$

that is exact for all polynomials of as high a degree as possible: that is determins α, a, b, and c.

$$\alpha = \sqrt{\frac{5}{7}},\ a = c = \frac{7}{25},\ b = \frac{8}{75}$$

Problem 32 [0pt] By the method of undetermined coefficients, derive a numerical integration formula of the form

$$\int_{-2}^{+2} |x|f(x)dx \approx Af(-1) + Bf(0) + Cf(1)$$

that is exact for polynomials of degree ≤ 2. Is it exact for polynomials of degree greater than 2?

$$A = \frac{8}{3},\ B = -\frac{4}{3},\ C = \frac{8}{3}$$

Yes. Exact for polynomials of degree ≤ 3

Problem 33 [0pt] If the function $f(x) = sin(100x)$ is to be approximated on the interval $[0,\pi]$ by an interpolating spline of degree1, how many knots are needed to ensure that $|S(x) - f(x)| < 10^{-8}$? (use equally spaced knots).

$$Knots \approx 1.57 \times 10^{10}$$

Problem 34 **[0pt]** Show that the derivative of a quadratic spline is a first-degree spline.

If S is piecewise quadratic then clearly $S^{'}$ is piecewise linear. If S is a quadratic spline then S has continuous first derivatives ($S \in C^1$). Hence $S^{'}$ is continuous. Therefore $S^{'}$ is piecewise linear and continuous.

Problem 35 **[0pt]** Do there exist a, b, c, and d such that the function

$$S(x) = \begin{cases} ax^3 + x^2 + cx & x \in [-1, 0] \\ bx^3 + x^2 + dx & x \in [0, 1] \end{cases}$$

is a natural cubic spline function that agrees with the absolute value function $|x|$ at the knots $-1, 0, 1$?

No

Problem 36 **[0pt]** Do there exist a, b, c, and d such that the function

$$S(x) = \begin{cases} -x & x \in [-10, -1] \\ ax^3 + bx^2 + cx + d & x \in [-1, 1] \\ x & x \in [1, 10] \end{cases}$$

is a natural cubic spline function?

no

Problem 37 **[0pt]** Determine the parameters a, b, c, d and e, such that S is a natural cubic spline:

$$S(x) = \begin{cases} a + b(x-1) + c(x-1)^2 + d(x-1)^3 & x \in [0, 1] \\ (x-1)^3 + ex^2 - 1 & x \in [1, 2] \end{cases}$$

$$a = -4, \; b = -6, \; c = -3, \; d = -1, \; e = -3$$

Problem 38 [0pt] Suppose $S(x)$ is an mth degree interpolating spline function over the interval $[a, b]$ with $n + 1$ knots $a = t_0 < t_1 < t_2 < \cdots < t_n = b$.

(a) How many conditions are needed to define $S(x)$ uniquely over $[a, b]$?

(b) How many conditions are defined by the interpolation conditions at the knots?

(c) How many conditions are defined by the continuity of the derivatives?

(d) How many additional conditions are needed so that the total equals the number in part a?

(a) $(m + 1)n$

(b) $2n$

(c) $(m - 1)(n - 1)$

(d) $m - 1$

Problem 39 [0pt] Determine the coefficients such that the function

$$S(x) = \begin{cases} x^2 + x^3 & x \in [0, 1] \\ a + bx + cx^2 + dx^3 & x \in [1, 2] \end{cases}$$

is a cubic spline and has properth $S_1'''(x) = 12$.

$$a = -1,\ b = 3,\ c = -2,\ d = 2$$