Intermediate Electromagnetism, UP 203

Notes on wave phase and polarization

1 Plane waves

A general plane wave in 1-D propagating in the positive $z-$ direction can be represented by $\psi(z,t) = \psi_0 \cos(kz - \omega t + \phi_0)$. Here ψ represents the amplitude (e.g., it can be density perturbation for a sound wave, electric field in $x-$ direction for a linearly polarized e.m. wave) and the phase $\phi = kz - \omega t + \phi_0$. Figure 1 shows the wave amplitude as a function of $(kz - \omega t)$ for different phase-lags ϕ_0. We can look at the wave amplitude as a function of time at a *fixed* z ($\psi[t] = \psi_0 \cos(kz_0 - \omega t + \phi_0) = \psi_0 \cos(\omega t - [kz_0 + \phi_0])$) or as a function of z at a fixed time ($\psi[z] = \psi_0 \cos(kz - \omega t_0 + \phi_0) = \psi_0 \cos(kz + [\phi_0 - \omega t_0])$). Thus, the wave in Figure 1 can be thought of as a function of kz with effective $\phi_0 = \phi_0 - \omega t_0$. Figure 1 can also be thought of as a function of ωt with effective $\phi_0 = -\phi_0 - kz_0$. If t_0 and z_0 are chosen to be zero (we are free to choose the origin of our coordinates), $\psi(t) = \psi_0 \cos(\omega t - \phi_0)$ and $\psi(z) = \phi_0 \cos(kz + \phi_0)$. Thus, if the wave is ahead of a reference wave in space, it will lag behind it in time. This is because the disturbance (ψ) at $z - v\Delta t$ ($v = \omega/k$ is the phase velocity) arrives at z after time Δt.

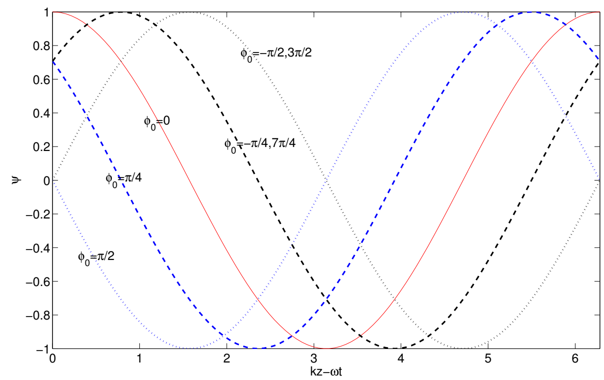

Figure 1: Wave amplitude (normalized to ψ_0) as a function of $kz - \omega t$ for different phase-lags ϕ_0; its a periodic function with period 2π. The wave with $\phi_0 = \pi/4$ lags behind the wave with $\phi_0 = 0$. The same is true if we treat ψ as a function of z at a fixed time. However, the wave with $\phi_0 = \pi/4$ will be ahead of the $\phi_0 = 0$ wave when looked as a function of time.

We can think of $\cos\phi$ as the real part of a complex phasor $e^{i\phi}$. Over a phase of 2π the phasor goes around a circle in the complex plane and its projection on real axis, $\cos\phi$, completes a full period. Its easier to work with phasors because phases are simply added; with $\sin/\cos$ we have to use trigonometric formulae. Moreover, integration/differentiation of an exponential function is trivial. So phasors are used everywhere in linear systems which support waves and oscillations. After applying all *linear* operations to phasors, the final solution is just the real part of the complex solution. We cannot directly use phasors for nonlinear operations; e.g., energy density ($\propto E^2$) in an e.m. wave.

Now let us consider interference between two waves traveling in the same direction (say z): $\psi_1 = \psi_{01} \cos(kz - \omega t)$ and $\psi_2 = \psi_{02} \cos(kz - \omega t + \Delta\phi)$, where $\Delta\phi$ is the phase difference between ψ_2 and ψ_1. This phase difference can correspond to the optical path difference at a distance x from the central fringe in Young's double hole experiment. There, we know that $\Delta\phi = 2\pi d \sin\theta/\lambda$ (d is spacing between holes and θ is the usual angle). It can also be due to a refractive thin film introduced in the path of the second wave; $\Delta\phi = 2\pi t(n-1)/\lambda$ (n is refractive index of the film and t is its thickness). For a small positive $\Delta\phi$, at z_0 a given wavefront from source 2 arrives a bit later compared to the corresponding wavefront of source 1 (verify this by drawing waves 1 & 2 as a function of time). In interference fringes we are looking at $\langle|\psi_1 + \psi_2|^2\rangle$ (where $\langle\rangle$ represents time average) at various locations in space. E.g., in Young's double hole experiment different locations on the screen ($z = D$) correspond to various *constant* phase differences of interfering waves.

2 Polarization

Unlike sound waves (but like waves on a stretched string), the $\vec{E}$ vector of e.m. waves lies in the plane perpendicular to $\vec{k}$. Velocity/density/pressure perturbation in a sound wave is either positive or negative, and does not have a direction/polarization. However, there are two independent components of a plane e.m. wave traveling along $z-$; $\vec{E} = E_{0x}\cos(kz-\omega t+\phi_{0x})\hat{x} + E_{0y}\cos(kz-\omega t+\phi_{0y})\hat{y}$ (both E_x and E_y are solutions of the wave equation and a linear combination should also be a solution, by superposition principle; of course $k = \omega n/c$). As outlined in class, a general polarization vector $(E_x\hat{x} + E_y\hat{y})$ traces out an ellipse in the $x-y$ plane. The representation of $E_x - E_y$ in the $x-y$ plane is a more compact way of visualizing the wave polarization rather than plotting E_x, E_y as a function of time. The polarization plot is analogous to the phase diagram in mechanics, where position and momentum are plotted against each other, instead of looking at position and momentum as functions of time.

A left circularly polarized wave corresponds to $E_x = E_y = E_0$ and $\phi_{0y} - \phi_{0x} = \pi/2$. Only relative phases matter because absolute phases just correspond to different starting locations for $\vec{E}$ in the $x-y$ plane. So, $\vec{E} = E_0 e^{i\phi}(\hat{x} + \hat{y}e^{i\pi/2})$, in phasor notation ($\phi = kz - \omega t + \phi_{0x}$). We can choose $kz + \phi_{0x} = 0$ and $\vec{E} = E_0[\hat{x}\cos\omega t + \hat{y}\cos(\omega t - \pi/2)]$. Thus $E_x/E_0 = \cos\omega t$ and $E_y/E_0 = \sin\omega t$; or, E_x and E_y trace out a circle in the $x-y$ plane. The polarization vector rotates counter-clockwise *when looking along* $-\hat{z}$ (see top panel of Fig. 2). We call this Left Circularly Polarized (LCP), consistent with Hecht's convention. The convention used in Hecht (this convention is what I was using in class) is opposite of the convention used in Ghatak/Feynman! Just to be safe in exams you can indicate the x, y, z axes when you show polarization vectors.

The bottom panels in Figure 2 show the temporal and polarization plots for a right elliptically polarized (REP) e. m. wave. The wave is given by $\vec{E} = E_0[0.5\hat{x}\cos\omega t + \hat{y}\cos(\omega t + \pi/4)] = E_0 e^{i\phi}(0.5\hat{x} + \hat{y}e^{-i\pi/4})$, where $\phi = kz - \omega t + \phi_{0x}$. The simplest way to trace the polarization ellipse (and its sense of rotation) in the $x-y$ plane is to look at the time variation of the $x-$ and $y-$ components of the e.m. wave and then locate the points corresponding to special points (such as corresponding to $E_x = 0$, $E_y = 0$, $E_x = E_{0x}$, $E_y = E_{0y}$) in the phase plane. Figure 2 shows two examples where special times in the temporal plot (a, b, c, d) are marked in the phase plot. All other polarization plots can be figured out in a similar way.

The orientation of the principal axes of the polarization ellipse, and its other properties, were indicated in class and is derived in both Hecht and Ghatak.

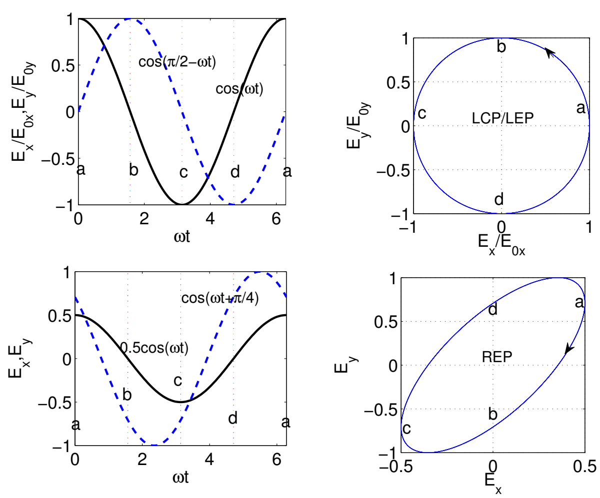

Figure 2: *Top Left Panel:* Amplitude as a function of time for E_x (solid black line) and E_y(blue dashed line) for LCP/LEP wave (LCP when $E_{0x} = E_{0y}$, LEP otherwise). *Top Right Panel:* The polarization plot in $x-y$ plane for LCP/LEP wave. *Bottom Left Panel:* Amplitude as a function of time for E_x (solid black line) and E_y(blue dashed line) for REP wave with $\phi_{0y} - \phi_{0x} = -\pi/4$ and $E_{0y} = 2E_{0x}$. *Bottom Right Panel:* The tip of the polarization vector traces out a clockwise ellipse. The labels a, b, c, d mark the corresponding locations in the temporal and polarization plots.