

The Drude Model

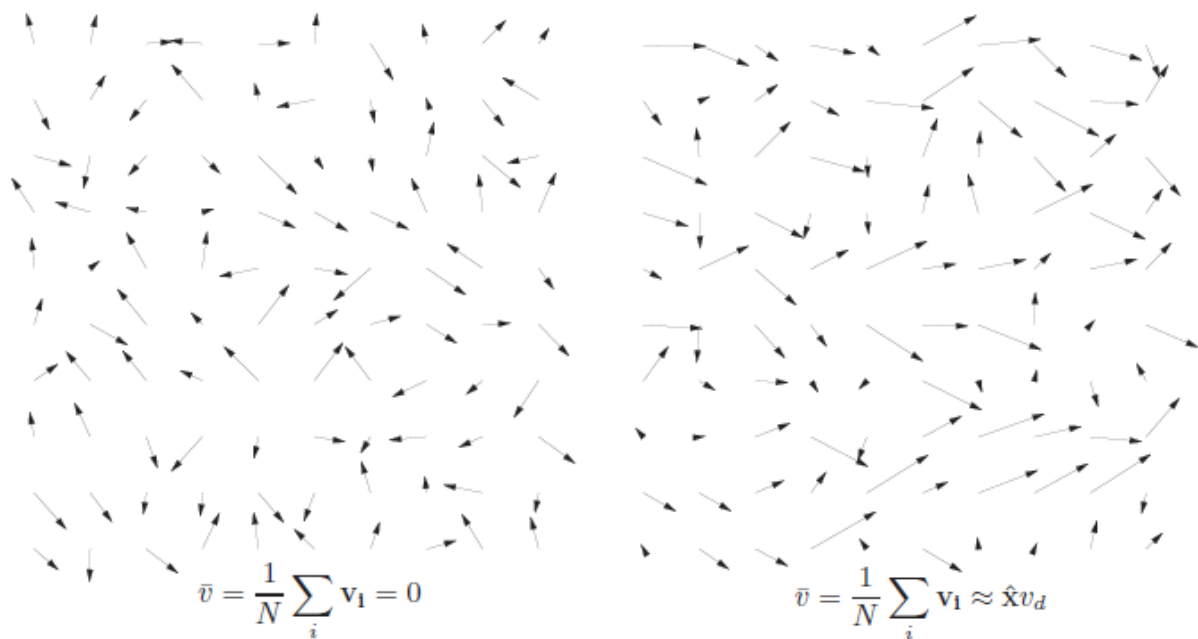
An intuitive picture of electron motion

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Physics 7B and Math 53 knowledge assumed

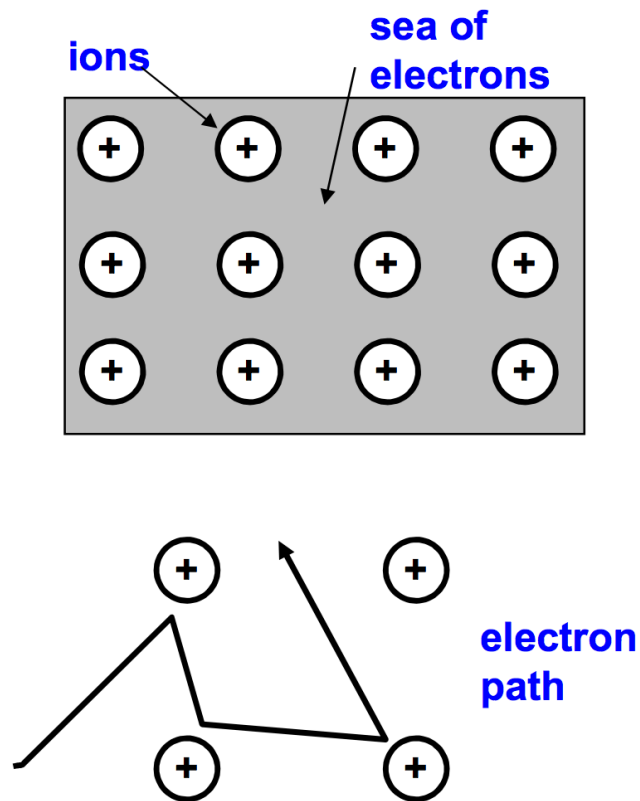
Images from Berkeley EE40 modules and Cornell ECE407 slides

In Module 1 you learned that the first trans-Atlantic telegraph cable suffered from severe signal distortion (more on distortion in cables if you take EE117 and EE210). Engineers at the time thought of the flow of electricity as an analogue of flowing water through pipes (interestingly the literal translations of the Chinese characters for voltage and current mean electric pressure and electric flow, respectively, suggesting that at the time the concepts were introduced into the language this analogy was commonplace). In some places today this analogy is still taught. Another analogy involving marbles or billiard balls also exists. However if you begin to probe these analogies carefully you will soon realize that they fall apart. For example, according to these analogies, information would move at whatever the speed of sound would be in a gas of electrons. From Module 2 you know that this isn't true. In that module you see slides with these pictures:



In this supplement we will look at these pictures more closely.

Consider a piece of conductor. In 1900 Paul Drude created a model which assumed that around the conductor's ions (atoms) are lots of electrons that can move freely about inside the conductor. He assumed that these electrons move according to Newton's laws until they scatter from 'colliding' with ions or defects, and after scattering an electron's momentum is totally random. At rest, electrons move around randomly from thermal energy and the average momentum of all the electrons is 0. This is illustrated in the picture below. Now let's see what happens when we apply an electric field.



First, some definitions for symbols we will be using later:

\vec{p} average momentum

\vec{v} average velocity

\vec{E} electric field

q charge

$-e$ electron charge

\vec{J} current density

m electron mass

n electron density (amount of electrons per unit space)

t time

$\frac{1}{\tau}$ scattering rate

In an incremental time interval Δt , Δt multiplied by the scattering rate is the scattering probability. If we apply Newton's law of the form 'the rate of change in momentum equals force' then we have $\Delta t/\tau$ probability of scattering which results in 0 average momentum (because the electron has a random momentum after scattering) and $1 - \Delta t/\tau$ probability of following Newton's law. The force on the electron is the Lorentz force $q\vec{E}$ (if there is no

magnetic field). We expand the expression for average momentum an incremental time step later as a Taylor series:

$$\vec{p}(t + \Delta t) = \left(1 - \frac{\Delta t}{\tau}\right) (\vec{p}(t) + q \vec{E}(t) \Delta t) + \frac{\Delta t}{\tau} (\vec{0})$$

$$\vec{p}(t) + \frac{d \vec{p}(t)}{dt} \Delta t + O(\Delta t^2) = \left(1 - \frac{\Delta t}{\tau}\right) (\vec{p}(t) + q \vec{E}(t) \Delta t)$$

$$\frac{d \vec{p}(t)}{dt} \Delta t + O(\Delta t^2) = -\vec{p}(t) \frac{\Delta t}{\tau} + q \vec{E}(t) \Delta t - q \vec{E}(t) \frac{\Delta t^2}{\tau}$$

$$\frac{d \vec{p}(t)}{dt} + O(\Delta t) = -\vec{p}(t) \frac{1}{\tau} + q \vec{E}(t) - q \vec{E}(t) \frac{\Delta t}{\tau}$$

$\lim (\Delta t \rightarrow 0) :$

$$\frac{d \vec{p}(t)}{dt} = q \vec{E}(t) - \frac{\vec{p}(t)}{\tau}$$

$O(x^2)$ means all the terms in the polynomial series with x^2 and higher powers. Notice the form of the last equation looks like that of an object in motion with linear drag (similar to an accelerating object with air or fluid resistance). Now we will look for a steady state solution by removing time dependencies and setting time derivatives to 0. Doing so in a drag motion equation solves for terminal velocity. Here instead it's the 'drift velocity':

$$0 = q \vec{E} - \frac{\vec{p}}{\tau}$$

$$\vec{p} = q \tau \vec{E} = -e \tau \vec{E}$$

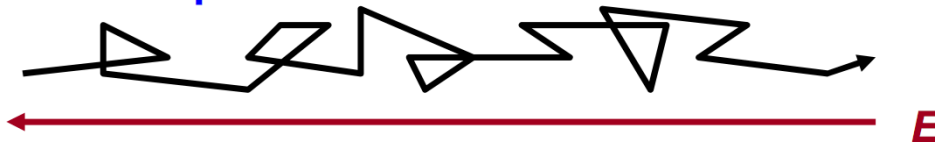
$$\vec{v} = \frac{\vec{p}}{m} = -\frac{e \tau \vec{E}}{m}$$

$$\vec{J} = n q \vec{v} = \frac{n e^2 \tau}{m} \vec{E}$$

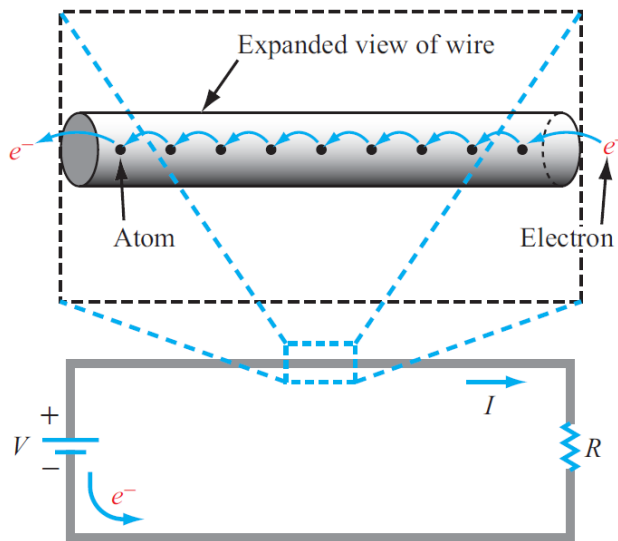
$$\vec{J} = \sigma \vec{E}$$

The last equation is Ohm's law in its less common 'differential' form. We defined current density as the charge density (electron density times electron charge) times its velocity, and conductivity $\sigma = ne^2\tau/m$. In the next part we will also define resistivity $\rho = 1/\sigma$. Now that average momentum is proportional to electric field, the picture of electron motion in the conductor looks like randomly moving electrons with an average (net) motion in one direction:

Electron path



Now we will assume that our conductive piece takes the shape of a cylinder with base area A and length L . Imagine that this is a length of wire:



We can derive the familiar Ohm's law by using the relations $E = -\nabla V$ and $I = AJ$:

$$\vec{J} = \sigma \vec{E}$$

$$\iiint \vec{J} \cdot d\vec{A} dL = \sigma \iiint \vec{E} \cdot d\vec{A} dL$$

$$IL = \sigma VA$$

$$V = I \frac{\rho L}{A}$$

$$V = IR$$

This model shows us that electric fields cause charges to move (electric current) by the Lorentz force, and electron scattering causes resistance. In Module 2 there is a discussion about whether 'electric information' travels at the speed of light in the wire and about 'aggregate motion of electrons'. Hopefully now it is clear to you that this is because electric fields move at the speed of light and move local electrons that are already present. The drift velocity in typical metals is in fact extremely slow – less than a millimeter per second. I hope this picture has given you a better physical intuition about the relationships between voltage, current, and resistance.