Kirchoff's Laws

The rules as corollaries of Maxwell's Equations
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Physics 7B and Math 53 knowledge assumed
Images from Berkeley EE40 modules and Cornell ECE303 slides

(This supplement is not discussed in discussion sections.)

$$\nabla \cdot \vec{D} = \rho$$

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

Above are the differential forms of Maxwell's Equations. You may have seen them in Physics 7B. In this supplement we will show that Kirchoff's circuit laws can be derived from Maxwell's Equations when certain assumptions are used. First to define some variables:

$\vec{E}$ electric field

$\vec{H}$ magnetic field

$\vec{D} = \epsilon \vec{E}$ "electric displacement"

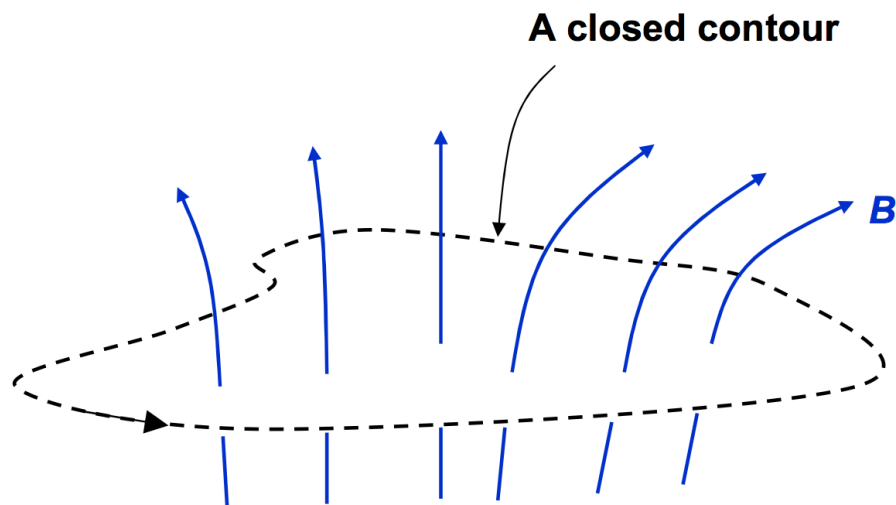$\vec{B} = \mu \vec{H}$ "magnetic induction"

$\vec{J}$ current density

ρ charge density

Φ magnetic flux

Q total charge

To get to KVL, we start with Faraday's Law and choose any surface in space bounded by a closed path:

A closed contour

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \text{("Faraday's Law")}$$

$$\int\int_s (\nabla \times \vec{E}) \cdot d\vec{s} = -\int\int_s \left(\frac{\partial \vec{B}}{\partial t}\right) \cdot d\vec{s}$$

$$\oint_\ell \vec{E} \cdot d\vec{\ell} = -\int\int_s \left(\frac{\partial \vec{B}}{\partial t}\right) \cdot d\vec{s}$$

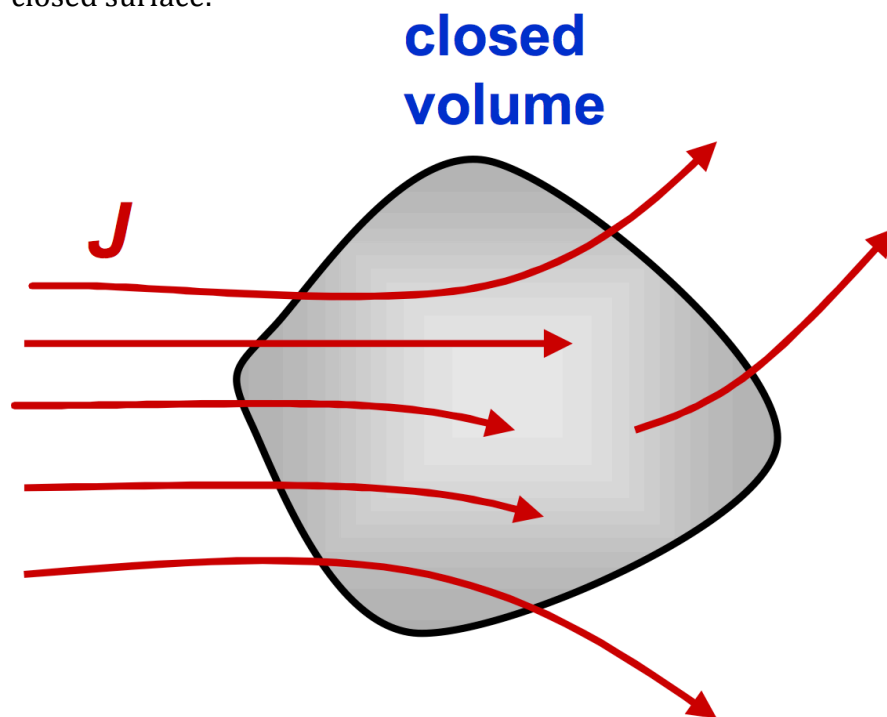$$\oint_\ell \vec{E} \cdot d\vec{\ell} = -\frac{\partial \Phi}{\partial t}$$

$$\sum_\ell V = -\frac{\partial \Phi}{\partial t} \qquad \text{("Lenz's Law")}$$

$$\sum_\ell V \simeq 0 \qquad \text{("Kirchoff's Voltage Law")}$$

Integrate Faraday's Law over the surface and use Stoke's Theorem to transform the electric field term into a contour integral around the closed path. The surface integral of magnetic induction is magnetic flux, and the contour integral of electric field is voltage, so the resulting equation is Lenz's Law. By dividing the closed path into discrete segments the contour integral effectively becomes a sum of the voltages across all the segments around the path. Finally, if we assume that the magnetic flux through the path changes negligibly with time, we end up with KVL.
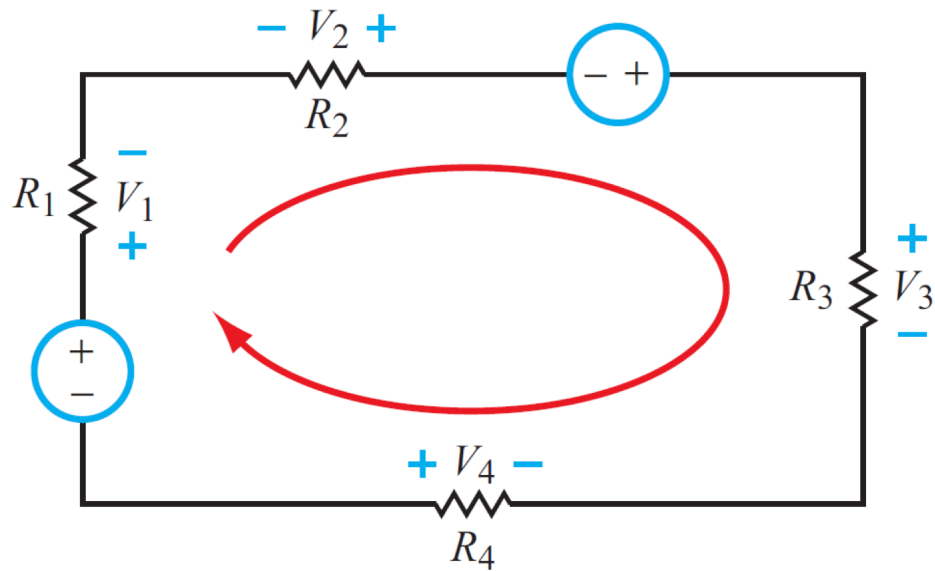
To get KCL, we start with Ampere's Law and choose any volume in space bounded by a closed surface:

closed volume

J

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \qquad \text{("Ampere's Law")}$$

$$\nabla \cdot \nabla \times \vec{H} = \nabla \cdot \left(\vec{J} + \frac{\partial \vec{D}}{\partial t} \right)$$

$$0 = \nabla \cdot \vec{J} + \nabla \cdot \frac{\partial \vec{D}}{\partial t}$$

$$\nabla \cdot \vec{J} = -\frac{\partial \rho}{\partial t} \qquad \text{("Continuity Equation")}$$

$$\iiint \nabla \cdot \vec{J} \, dv = -\iiint \frac{\partial \rho}{\partial t} \, dv$$

$$\oiint \vec{J} \cdot d\vec{s} = -\iiint \frac{\partial \rho}{\partial t} \, dv$$

$$\oiint \vec{J} \cdot d\vec{s} = -\frac{\partial Q}{\partial t}$$

$$\sum_{s} I = -\frac{\partial Q}{\partial t}$$

$$\sum_{s} I \simeq 0 \qquad \text{("Kirchoff's Current Law")}$$

Take the divergence of Ampere's Law and use the vector identity $\nabla \cdot \nabla \times \vec{v} = 0$ to eliminate the magnetic field term. Substitute Gauss's Law into the electric displacement term to arrive at the Continuity Equation. Integrate this equation over the volume and use the Divergence Theorem to transform the current density term into a surface integral over the closed surface. The volume integral of charge density is total charge inside the volume, and the surface integral of current density is current. By noting where the closed surface cuts through areas with current, such as conductors, the surface integral effectively becomes a sum of the currents in all the conductors passing through the surface. Since closed surface normal vectors face outward, the sum assigns all current directions to be flowing away from the surface. Finally, if we assume that the total charge in the volume changes negligibly with time, we end up with KCL.
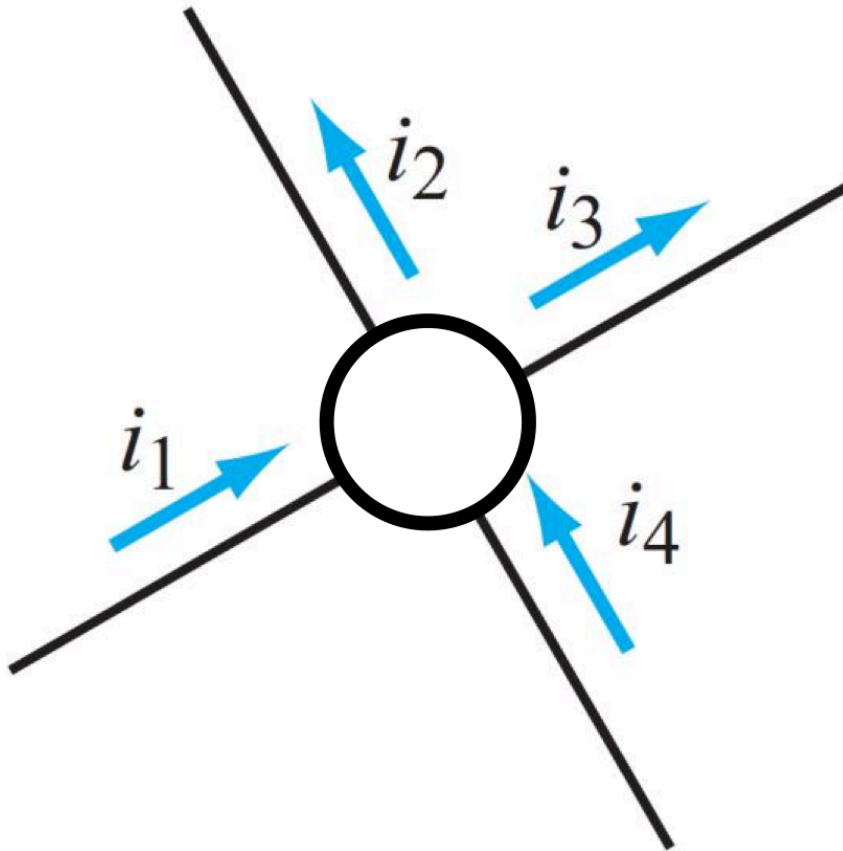
For certain choices of closed surfaces (those enclosing a voltage source), you get a 'supernode', and likewise for certain choices of closed paths (those enclosing a current source), you get a 'supermesh'.

Let's look at when are Kirchoff's Laws would be violated. Notice that both laws assume the time rate of change of something is approximately 0. This means they are true in the 'static' or 'low frequency' limit. The circuits that you will deal with in this course work under this limit. However, it's interesting to note that KCL is violated **inside** the capacitor because charge can build up on the capacitor plates, so if part of your closed surface passes between the plates the net charge changes significantly with time. However if you consider the capacitor as a whole 'lumped element' and include both plates inside the closed surface, the charges on opposite plates cancel out and KCL will still work. Similarly, KVL is violated **inside** the inductor because a significant magnetic flux passes through the coil, so if your closed path runs along the coil's wiring a voltage drop occurs even across the perfect metal wire. However if you consider the inductor as a whole 'lumped element' and recognize its total voltage as the induced electromotive force due to Lenz's Law, KVL will still work.

Finally, a 'high frequency' circuit where Kirchoff's Laws are violated is one in which the time it takes light to travel through the length of your circuit is longer than the period of oscillation (reciprocal of frequency). If a typical circuit you build on a breadboard is 30 cm on the longest loop, this frequency limit would be 1 GHz. To learn more see EE117, EE210, and courses about microwave and radio frequency electronics.