Math 312: Lecture 19

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- Reading for today is section 11.
- We are building on the notion of *subsequence* of a given sequence.
- The important point is that basically *every* sequence has a convergent subsequence, i.e. has a subsequential limit.

The Bolzano-Weierstrass Theorem

- Suppose that a sequence (x_n) is given. Recall that *subsequence* of (x_n) is obtained by "selecting terms" from (x_n).
- We defined a *subsequential limit* to be a number which is the limit of some *convergent* subsequence of (x_n).
- We saw several examples of sequences that have *more than one* subsequential limit. But could there be a sequence without *any* subsequential limits?

The answer "no" is another manifestation of the Completeness Axiom. It is called the *Bolzano-Weierstrass theorem*

Bolzano and Weierstrass

Monotone subsequences

Theorem

Every sequence has a monotone subsequence.

What is usually called the Bolzano-Weierstrass theorem is a corollary of this.

Corollary

Every bounded sequence has a convergent subsequence (to a finite limit).

Proof.

A monotone subsequence of a bounded sequence is bounded (because the parent sequence is). And a bounded monotone sequence converges (by an earlier result). □

Proof of the Monotone Subsequence Theorem

Consider a general sequence (x_n).

Definition

We'll say that n is a *peak* for the sequence if, for all $m \geqslant n$, $x_n \geqslant x_m$ (i.e. the n'th term is "as good as it's going to get").

We consider two cases:

- Case 1: there are infinitely many peaks.
- Case 2: there are finitely many peaks.

Proof in Case 1

In Case 1, there are infinitely many peaks. Let the peaks be $n_1, n_2, n_3, \ldots$ in increasing order. Then $x_{n_1}, x_{n_2}, x_{n_3}, \ldots$ forms a subsequence.

Because n_1 is a peak and $n_1 \leqslant n_2$, $x_{n_1} \geqslant x_{n_2}$. Because n_2 is a peak and $n_2 \leqslant n_3$, $x_{n_2} \geqslant x_{n_3}$, and so on.

Thus the subsequence (x_{n_k}) is monotone decreasing.

Proof in Case 2

In Case 2, there are only finitely many peaks. Thus there is a last peak, call it N.

Let $n_1 = N + 1$. Then n_1 is not a peak, so there is $n_2 > n_1$ such that $x_{n_1} < x_{n_2}$. Also, n_2 is not a peak, so there is $n_3 > n_2$ such that $x_{n_2} < x_{n_3}$. Keep going in this way, defining n_{k+1} to be such that $x_{n_{k+1}} > x_{n_k}$, using the fact that n_k is not a peak.

Then the subsequence (x_{n_k}) is monotone increasing.