Linear Programming

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Introduction to Optimization

Optimization is driven by competition, quality assurance, cost of production and operations, and the success of the business enterprise. Ignoring the practice of optimization in *not* an option during current times

Definition of Optimization, or Mathematical Programming: one seeks to minimize or maximize a function by systematically choosing the values of real or integer variables from within an allowed set; simply speaking: the best result under the circumstances. Optimization is practiced through software programs and requires significant computer resources.

Definition of the decision variable: An unknown quantity representing a decision that needs to be made. A quantity the optimization engine has to determine. It is known as a controllable input variable.

Example: *XY* is decision variable representing an amount of a product needs to be delivered from a warehouse *X* to retail store *Y*.

People use the word "optimization" to describe the primary quality of our work or endeavor. “Optimization” is probably most used or abused term in advertising and presentations. It's fine and it shows making the best choice is our perpetual desire among all of us

Examples of Optimization Research and Applications in CAAM:

- NASA: aerospace design to avoid to lower the tremendous cost associated with carrying unnecessary weight in space vehicles; also, optimal control of space stations (using spinning gyroscopes to maneuver space stations instead of rockets)
- Optimization-based Signal / Image Processing: image denoising and deblurring; object recognition.
- Portfolio optimization.
- Many more …

Until recently, optimization was attempted only in those situations where there were significant penalties for generic (non-optimal) designs.

- Causes: sophisticated optimization requires advanced mathematics both modeling and solving; opt. demands large computational resources, used to exist only at large national laboratories.
- Results: Most of the everyday products around you were designed without regard to optimization; even the new generation of replacement products, like the car, the house, the desk, the computer, redesigned and manufactured today are not designed optimally in one sense or another

Personally, you would definitely explore procedures to optimize

- your investments by tailoring your portfolio
- your travel time by appropriately choosing your routes and destinations
- your commuting time by choosing your time and roads

- expenditure for living by choosing your day store for shopping
- your learning by attending a class or not (instead, read the textbook and notes by yourself)
- you can buy books, read articles, use software to perform these various optimization

This justifies the necessity of learning mathematical optimization tools

A Linear Programming Example

In the problem, a company manufactures two iPod player models, both with 3.5" LCD but have different memory capacities

- 16GB - 2x8GB chips
- 8GB - 1x8GB chip

Weekly resources are limited to

- 800 units of 3.5" antiglare LCDs
- 1000 units of 8GB memory chips
- 50 hours of total labor time. It takes 3 minutes of labor for each 16GB player, and 4 minutes of labor for each 8GB player.

For marketing reasons,

- Total production cannot exceed 700
- Number of 16GB players cannot exceed number of 8GB players by more than 350

Profit, while remaining within the marketing guidelines, can be computed as

- $16 each 16GB player
- $10 each 8GB player

The company’s current weekly production plan consists of 450 16GB players and 100 8GB players, make a profit of 16*450 + 10*100 = $8200. Management is seeking a production schedule that will *increase the company's profit*. A linear programming model can provide an insight and an intelligent solution to this problem.

The elements of a linear program include decision variables, objective function, and (optional) constraints. For the iPod manufacturing problem, the decision variables are

- x1: weekly produced units of 16GB players
- x2: weekly produced units of 8GB players

The objective is to maximize weekly profit 16*x1+10*x2

There are several constraints:

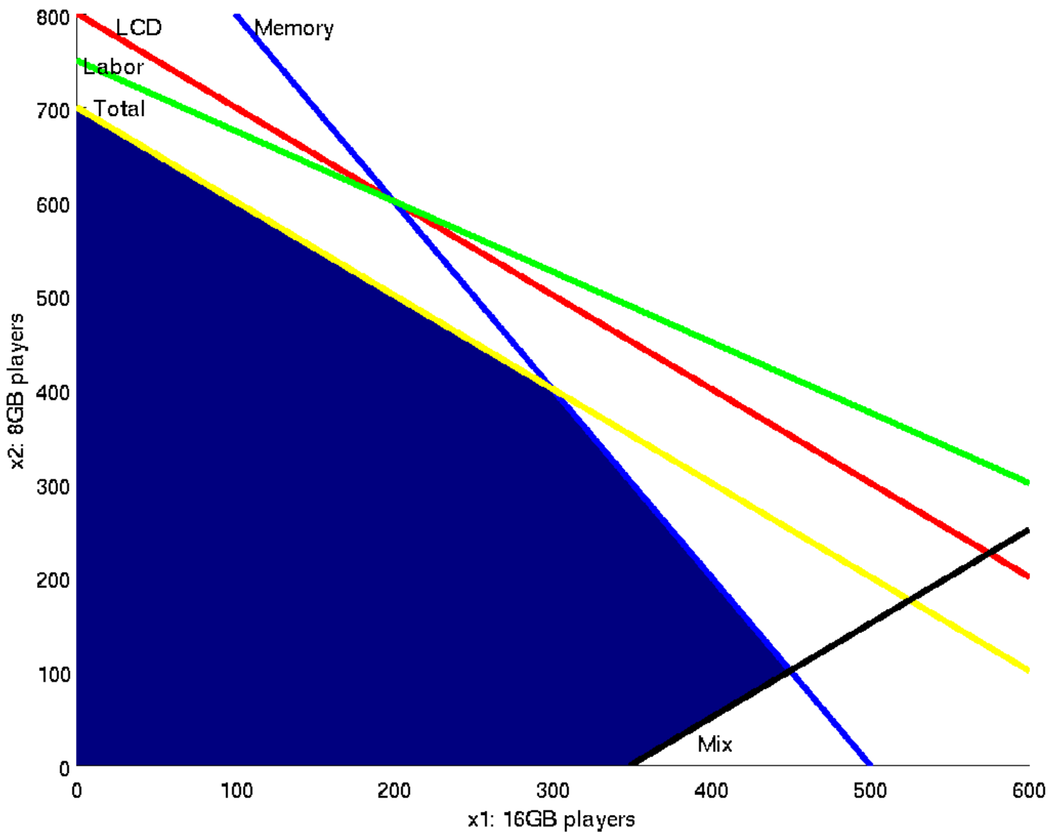
- x1>=0, x2>=0. no buy-back! We cannot buy 8GB players, open them, and use the LCDs and memory units to produce 16GB players
- LCD: x1+x2<=800
- memory: 2*x1+x2<=1000
- labor: 3*x1+4*x2<=3000
- marketing
- total: x1+x2<=700
- mix: x1-x2<=350

The so-called data of a linear optimization include the coefficients of the linear objective function and the numbers on the right-hand sides of the constraints.

Graphical Optimization

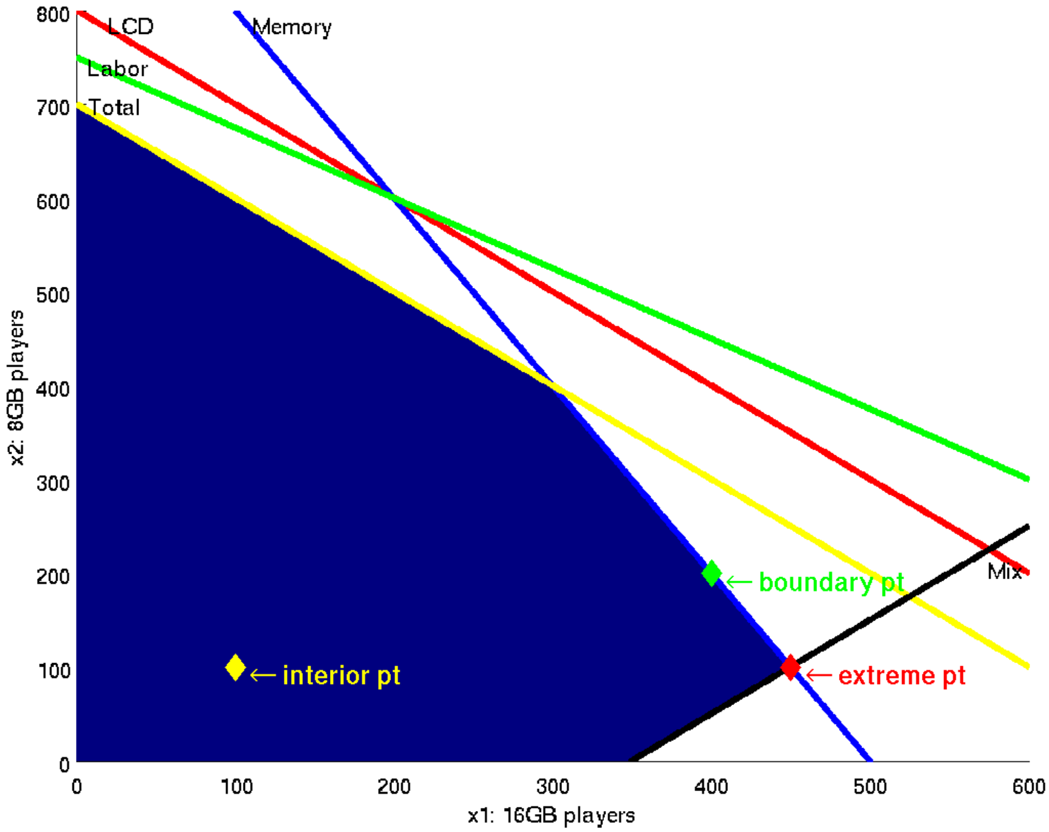
Using graphs, we can represent all of the constraints, the objective function, the three types of feasible points, and find an optimal solution.

The feasible region is the set of all points that satisfy all constraints

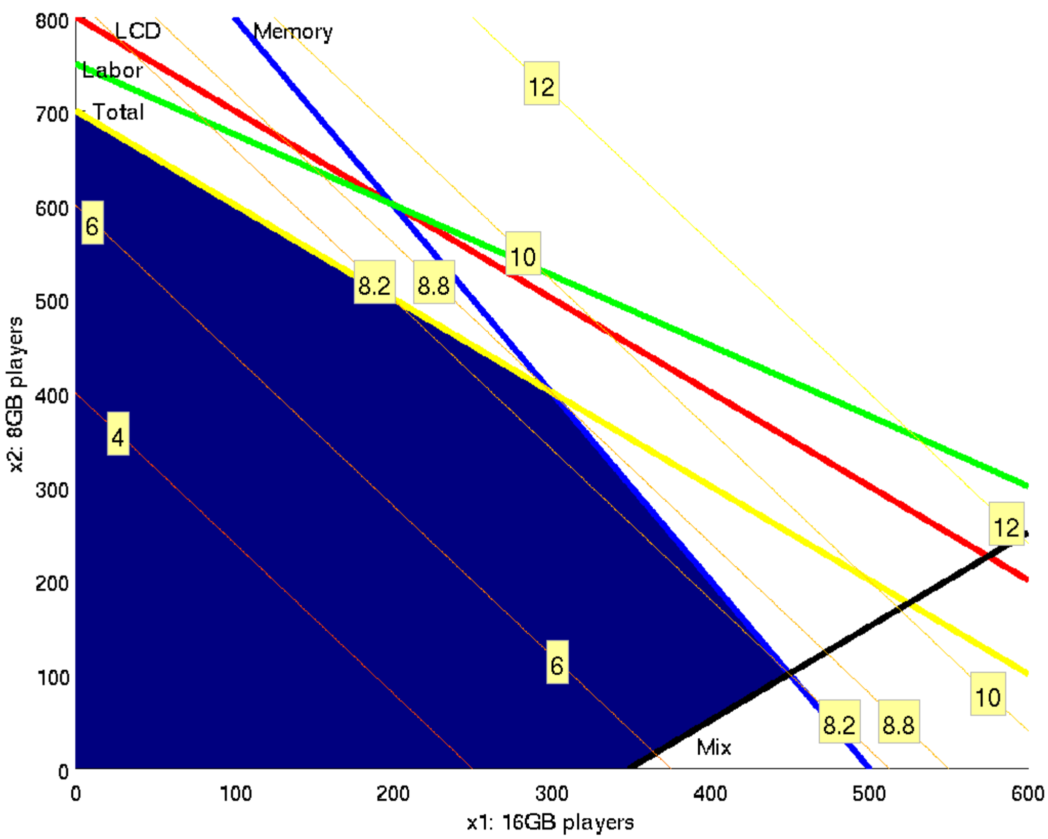


There are three types of feasible points

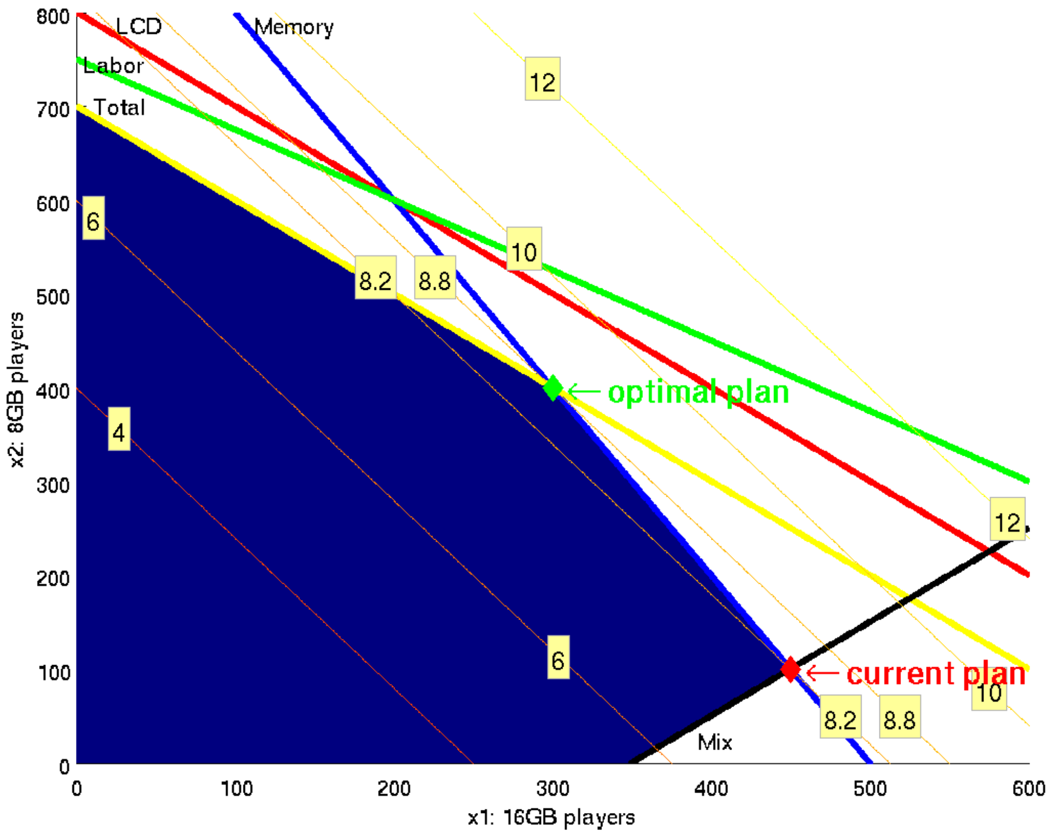
- Interior point
- Boundary point
- Extreme point



Because the objective function is linear, objective values are shown by parallel level lines.



The optimal solution is one that maximizes the objective function. In this example, the optimal solution is x1=300, x2=400. The optimal objective value: 16*x1+10*x2=$8800. This solution is better than the current plan x1=450, x2=100 in terms of profit by $600. Even though producing a 16GB player is more profitable than producing a 8GB player, the optimal plan produces 100 more 8GB players than 16 players because it utilizes all memory and total allowed production.



Linear Optimization in MATLAB

We can solve linear programs in MATLAB by calling Optimization Toolbox function: linprog.

linprog takes one of the standard forms of linear programs:

*Min f'*x*

Subject to

*A*x <= b*

*Aeq*x = beq*

lb <= x <= ub.

Our linear programming formulation is

*Max 16*x1 +10*x2* *(profit)*

Subject to

x1+x2 <= 800 *(LCD)*

*2*x1+x2 <= 1000* *(memory)*

*3*x1+4*x2 <=3000* *(labor)*

x1+x2 <= 700 *(market: total)*

x1-x2 <= 350 *(market: mix)*

x1 >= 0

x2 >= 0

The syntax of linprog is [outputs] = linprog(f,A,b,Aeq,beq,lb,ub,start point,options). It is important to note that minimizing an objective function is equivalent to maximizing it negative. See the MATLAB program for a demo.

It is important to know some general results of linear programming (LP).

1. An LP may have multiple optimal solutions. Try updating profit of 8GB plays to $8.
2. An LP may have no optimal solution, for example, under the following two cases:
   a) The feasible region is empty. Try adding another constraint: an order to fill requires at least 500 16GB players.
   b) The feasible region is nonempty but objective can become infinitely large. Try removing all but the nonnegativity and mix constraints.
3. If an LP has an optimal solution and an extreme point, then there exists an optimal solution at an extreme point. Try varying the profit rates for the two players

Advanced topic: Sensitivity Analysis

After an LP is solved, one often asks the question: Will the optimal solution change when input parameters change? Reasons of asking include

- Parameters used were only best estimates
- Parameters change over time
- "What-if" analysis help make operational decisions

For instance, for manufacturing MP4 players, we may ask

- Is it worth buying extra LCDs at a higher price? If yes, what is an acceptable price?
- Is it worth paying employees to work over hours? If yes, what is an acceptable rate?
- What is the most critical resource?
- Many more ......

Sensitivity analysis also study how the optimal solution changes after adding or dropping a constraint, adding or dropping a decision variable (e.g., an MP4 player model)