CS 242 Lecture 2

- Topics:
  Sections 1.1 and 1.2

- In-class assignment # 1
  Exercises from Section 1.2

- Quiz # 1

Logic

- Crucial for mathematical reasoning
- Used for designing electronic circuitry

- Logic is a system based on propositions.
- A proposition is a statement that is either true or false (not both).
- We say that the truth value of a proposition is either true (T) or false (F).

- Corresponds to 1 and 0 in digital circuits

Examples of propositions

EXAMPLE 1 All the following declarative sentences are propositions.

1. Washington, D.C., is the capital of the United States of America.
2. Toronto is the capital of Canada.
3. $1 + 1 = 2$.
4. $2 + 2 = 3$.

Propositions 1 and 3 are true, whereas 2 and 4 are false. ◀

Some sentences that are not propositions are given in Example 2.

EXAMPLE 2 Consider the following sentences.

1. What time is it?
2. Read this carefully.
3. $x + 1 = 2$.
4. $x + y = z$.

Sentences 1 and 2 are not propositions because they are not declarative sentences. Sentences 3 and 4 are not propositions because they are neither true nor false. Note that each of sentences 3 and 4 can be turned into a proposition if we assign values to the variables. We will also discuss other ways to turn sentences such as these into propositions in Section 1.4. ◀

Propositional logic

- The area of logic that deals with propositions is called the propositional calculus or propositional logic.
- It was first developed systematically by the Greek philosopher Aristotle more than 2300 years ago.

Combining Propositions

- one or more propositions can be combined to form a single compound proposition.
- George Boole in 1854 in his book *The Laws of Thought introduced* various operations and studied their properties.
- We denote the basic (atomic) propositions by letters such as p, q, r, s, and introduce several logical operators.

Logical Operators (Connectives)

We will examine the following logical operators:

- Negation (NOT)
- Conjunction (AND)
- Disjunction (OR)
- Exclusive or (XOR)
- Implication (if – then)
- Biconditional (if and only if)

A main goal of propositional logic is to determine the truth of compound propositions.

Negation (NOT)

Unary Operator, Symbol: $\neg$

P	$\neg$P
true	false
false	true

Negation Example

EXAMPLE 4 Find the negation of the proposition

"Vandana's smartphone has at least 32GB of memory"

and express this in simple English.

Solution: The negation is

"It is not the case that Vandana's smartphone has at least 32GB of memory."

This negation can also be expressed as

"Vandana's smartphone does not have at least 32GB of memory"

or even more simply as

"Vandana's smartphone has less than 32GB of memory." ◀

Conjunction (AND)

Binary Operator, Symbol: $\wedge$

P	Q	$P \wedge Q$
true	true	true
true	false	false
false	true	false
false	false	false

Conjunction - Example

EXAMPLE 5 Find the conjunction of the propositions p and q where p is the proposition "Rebecca's PC has more than 16 GB free hard disk space" and q is the proposition "The processor in Rebecca's PC runs faster than 1 GHz."

Solution: The conjunction of these propositions, $p \wedge q$, is the proposition "Rebecca's PC has more than 16 GB free hard disk space, and the processor in Rebecca's PC runs faster than 1 GHz." This conjunction can be expressed more simply as "Rebecca's PC has more than 16 GB free hard disk space, and its processor runs faster than 1 GHz." For this conjunction to be true, both conditions given must be true. It is false, when one or both of these conditions are false. ◀

Disjunction (OR)

Binary Operator, Symbol: ∨

P	Q	P∨Q
true	true	true
true	false	true
false	true	true
false	false	false

Exclusive Or (XOR)

Binary Operator, Symbol: $\oplus$

P	Q	$P \oplus Q$
true	true	false
true	false	true
false	true	true
false	false	false

Implication (if - then)

Binary Operator, Symbol: $\rightarrow$

P	Q	P$\rightarrow$Q
true	true	true
true	false	false
false	true	true
false	false	true

($\rightarrow$ is the most confusing logical operation. Make sure you understand this truth table.)

Many ways to express a conditional statement $p \rightarrow q$

"if p, then q"
"if p, q"
"p is sufficient for q"
"q if p"
"q when p"
"a necessary condition for p is q"
"q unless $\neg p$"

"p implies q"
"p only if q"
"a sufficient condition for q is p"
"q whenever p"
"q is necessary for p"
"q follows from p"

Converse, contrapositive

- Converse of $p \rightarrow q$:

  $q \rightarrow p$

- Contrapositive of $p \rightarrow q$:

  $(\neg q) \rightarrow (\neg p)$

Example of converse and contrapositive

EXAMPLE 9 What are the contrapositive, the converse, and the inverse of the conditional statement

"The home team wins whenever it is raining?"

Extra Examples

Solution: Because "q whenever p" is one of the ways to express the conditional statement $p \to q$, the original statement can be rewritten as

"If it is raining, then the home team wins."

Consequently, the contrapositive of this conditional statement is

"If the home team does not win, then it is not raining."

The converse is

"If the home team wins, then it is raining."

Biconditional (if and only if)

Binary Operator, Symbol: $\leftrightarrow$

P	Q	$P \leftrightarrow Q$
true	true	true
true	false	false
false	true	false
false	false	true

Making larger compound statements

Just as we can use operators like + and x many times and create compound arithmetic expressions (such as 5 + 4 x (3 + 2) etc.) we can create compound statements in propositional logic using the operators V, → , ∧ etc.

Examples:

- $(\neg P)\vee(\neg Q)$
- $((P\vee(\neg Q)) \rightarrow R) \rightarrow (\neg Q)$

Note that order of evaluations need to be specified in order to interpret expressions.

Is 3 + 4 x 5 = (3 + 4) x 5 or = 3 + (4 x 5)?

Order of evaluation (precedence rules)

TABLE 8
Precedence of Logical Operators.

Operator	*Precedence*
$\neg$	1
$\wedge$	2
$\vee$	3
$\rightarrow$	4
$\leftrightarrow$	5

Thus, the expression ~p V q
Denotes (~p) V q, not ~(p V q).

Similarly, p V q $\wedge$ r denotes
P V (q $\wedge$ r), not (p V q) $\wedge$ r

Exercise: Show the fully parenthesized form of
p → q V ~r

Binary vs. boolean

In many computer science applications, T (true) is denote by 1 and F (false) by 0.

0 and 1 are the basic 'digits' in binary system.

We can extend Boolean operations such as ~, V etc. to binary strings.

Example: 01001 V 00100 = 01101 etc.

More examples of bitwise Boolean operations

EXAMPLE 13 Find the bitwise *OR*, bitwise *AND*, and bitwise *XOR* of the bit strings 01 1011 0110 and 11 0001 1101. (Here, and throughout this book, bit strings will be split into blocks of four bits to make them easier to read.)

Solution: The bitwise *OR*, bitwise *AND*, and bitwise *XOR* of these strings are obtained by taking the *OR*, *AND*, and *XOR* of the corresponding bits, respectively. This gives us

```
01 1011 0110
11 0001 1101
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11 1011 1111   bitwise OR
01 0001 0100   bitwise AND
10 1010 1011   bitwise XOR
```

Statements and Operators

Evaluation of a compound expression

P	Q	$\neg P$	$\neg Q$	$(\neg P) \vee (\neg Q)$
true	true	false	false	false
true	false	false	true	true
false	true	true	false	true
false	false	true	true	true

Statements and Operations

Statements and operators can be combined in any way to form new statements.

P	Q	$P \wedge Q$	$\neg (P \wedge Q)$	$(\neg P) \vee (\neg Q)$
true	true	true	false	false
true	false	false	true	true
false	true	false	true	true
false	false	false	true	true

Equivalent Statements

P	Q	$\neg(P \wedge Q)$	$(\neg P) \vee (\neg Q)$	$\neg(P \wedge Q) \leftrightarrow (\neg P) \vee (\neg Q)$
true	true	false	false	true
true	false	true	true	true
false	true	true	true	true
false	false	true	true	true

The statements $\neg(P \wedge Q)$ and $(\neg P) \vee (\neg Q)$ are logically equivalent, because $\neg(P \wedge Q) \leftrightarrow (\neg P) \vee (\neg Q)$ is always true.

Tautologies and Contradictions

A tautology is a statement that is always true.

Examples:

- $R \vee (\neg R)$
- $\neg(P \wedge Q) \leftrightarrow (\neg P) \vee (\neg Q)$

If $S \rightarrow T$ is a tautology, we write $S \Rightarrow T$.

If $S \leftrightarrow T$ is a tautology, we write $S \Leftrightarrow T$.

Tautologies and Contradictions

A contradiction is a statement that is always false.

Examples:

- $R \wedge (\neg R)$
- $\neg(\neg(P \wedge Q) \leftrightarrow (\neg P) \vee (\neg Q))$

The negation of any tautology is a contra-diction, and the negation of any contradiction is a tautology.

Exercises

We already showed that the following is a tautology:

$$\neg(P \wedge Q) \Leftrightarrow (\neg P) \vee (\neg Q)$$

Similarly we can show:

$$\neg(P \vee Q) \Leftrightarrow (\neg P) \wedge (\neg Q)$$

Exercises

11. Let p and q be the propositions

p : It is below freezing.
q : It is snowing.

Write these propositions using p and q and logical connectives (including negations).

a) It is below freezing and snowing.
b) It is below freezing but not snowing.
c) It is not below freezing and it is not snowing.
d) It is either snowing or below freezing (or both).
e) If it is below freezing, it is also snowing.
f) Either it is below freezing or it is snowing, but it is not snowing if it is below freezing.
g) That it is below freezing is necessary and sufficient for it to be snowing.

Exercises

11. Let p and q be the propositions

p : It is below freezing.

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Write these propositions using p and q and logical connectives (including negations).

a) It is below freezing and snowing.
b) It is below freezing but not snowing.
c) It is not below freezing and it is not snowing.
d) It is either snowing or below freezing (or both).
e) If it is below freezing, it is also snowing.
f) Either it is below freezing or it is snowing, but it is not snowing if it is below freezing.
g) That it is below freezing is necessary and sufficient for it to be snowing.

a) $p \wedge q$
b) $p \wedge \sim q$
c) $\sim p \wedge \sim q$
d) $p \vee q$
e) $p \rightarrow q$
f) $(p \vee q) \wedge (p \rightarrow \sim q)$
g) $p <-> q$

Exercises

12. Let p, q, and r be the propositions

p : You have the flu.
q : You miss the final examination.
r : You pass the course.

Express each of these propositions as an English sentence.

a) $p \rightarrow q$
b) $\neg q \leftrightarrow r$
c) $q \rightarrow \neg r$
d) $p \vee q \vee r$
e) $(p \rightarrow \neg r) \vee (q \rightarrow \neg r)$
f) $(p \wedge q) \vee (\neg q \wedge r)$

Exercises

12. Let p, q, and r be the propositions

p : You have the flu.
q : You miss the final examination.
r : You pass the course.

Express each of these propositions as an English sentence.

a) $p \to q$
b) $\neg q \leftrightarrow r$
c) $q \to \neg r$
d) $p \vee q \vee r$
e) $(p \to \neg r) \vee (q \to \neg r)$
f) $(p \wedge q) \vee (\neg q \wedge r)$

a) If you have the flu, you miss the final examination.

b) You pass the course if and only if you don't miss the final examination.

f) You have the flu and miss the final examination, or you don't miss the final examination and pass the course.

Exercises

22. Write each of these statements in the form "if p, then q" in English. [*Hint:* Refer to the list of common ways to express conditional statements provided in this section.]

a) It is necessary to wash the boss's car to get promoted.
b) Winds from the south imply a spring thaw.
c) A sufficient condition for the warranty to be good is that you bought the computer less than a year ago.
d) Willy gets caught whenever he cheats.
e) You can access the website only if you pay a subscription fee.
f) Getting elected follows from knowing the right people.
g) Carol gets seasick whenever she is on a boat.

31. Construct a truth table for each of these compound propositions.

a) $p \wedge \neg p$
b) $p \vee \neg p$
c) $(p \vee \neg q) \rightarrow q$
d) $(p \vee q) \rightarrow (p \wedge q)$
e) $(p \rightarrow q) \leftrightarrow (\neg q \rightarrow \neg p)$
f) $(p \rightarrow q) \rightarrow (q \rightarrow p)$

Solutions will be worked out in class by the instructor.

Exercises

30. How many rows appear in a truth table for each of these compound propositions?

a) $(q \to \neg p) \vee (\neg p \to \neg q)$

b) $(p \vee \neg t) \wedge (p \vee \neg s)$

c) $(p \to r) \vee (\neg s \to \neg t) \vee (\neg u \to v)$

d) $(p \wedge r \wedge s) \vee (q \wedge t) \vee (r \wedge \neg t)$

Exercises

30. How many rows appear in a truth table for each of these compound propositions?

a) $(q \rightarrow \neg p) \vee (\neg p \rightarrow \neg q)$

b) $(p \vee \neg t) \wedge (p \vee \neg s)$

c) $(p \rightarrow r) \vee (\neg s \rightarrow \neg t) \vee (\neg u \rightarrow v)$

d) $(p \wedge r \wedge s) \vee (q \wedge t) \vee (r \wedge \neg t)$

Answers:

a) 4 b) 8 c) 64 d) 32

Exercises

***49.** The nth statement in a list of 100 statements is "Exactly n of the statements in this list are false."

a) What conclusions can you draw from these statements?

b) Answer part (a) if the nth statement is "At least n of the statements in this list are false."

c) Answer part (b) assuming that the list contains 99 statements.

This one is slightly tricky – as indicated by a *. Solution for (a) is in the next slide.

Exercises

***49.** The *n*th statement in a list of 100 statements is "Exactly *n* of the statements in this list are false."

a) What conclusions can you draw from these statements?

b) Answer part (a) if the *n*th statement is "At least *n* of the statements in this list are false."

c) Answer part (b) assuming that the list contains 99 statements.

a) clearly, exactly one of the hundred statements is true. (Why? Suppose two of them are true. One of them says "exactly j statements are false" and the other says "exactly k statements are false" and k != j. This is impossible.)
So which one is the correct one? You can check that the 99^{th} one is because it says "exactly 99 statements are false" which is correct because all other 99 are false.

In-class assignment

Please work on the following problems:

Problems 4, 26, 27, 38, 43, 49 b

After you complete it, make sure to write your name and submit it to the instructor.