Lecture 3 Jan 22, 2014

Topics covered: (Sec 1.2)

- Applications of propositional logic
  - Combinational logic circuits
  - Logic puzzles
  - Specifications using propositional logic
  - Propositional logic in web search

- Propositional equivalence (Sec 1.3)
  - Proof of equivalence
  - Satisfiability testing

- Predicate logic (basic definitions) Sec 1.4

Propositional Logic for expressing requirements or conditions

EXAMPLE 1 How can this English sentence be translated into a logical expression?

"You can access the Internet from campus only if you are a computer science major or you are not a freshman."

Extra Examples

Solution: There are many ways to translate this sentence into a logical expression. Although it is possible to represent the sentence by a single propositional variable, such as p, this would not be useful when analyzing its meaning or reasoning with it. Instead, we will use propositional variables to represent each sentence part and determine the appropriate logical connectives between them. In particular, we let a, c, and f represent "You can access the Internet from campus," "You are a computer science major," and "You are a freshman," respectively. Noting that "only if" is one way a conditional statement can be expressed, this sentence can be represented as

$$a \to (c \lor \neg f).$$

◀

Propositional logic for specification

Exercise

5. You are eligible to be President of the U.S.A. only if you are at least 35 years old, were born in the U.S.A, or at the time of your birth both of your parents were citizens, and you have lived at least 14 years in the country. Express your answer in terms of e: "You are eligible to be President of the U.S.A.," a: "You are at least 35 years old," b: "You were born in the U.S.A," p: "At the time of your birth, both of your parents where citizens," and r: "You have lived at least 14 years in the U.S.A."

Propositional logic for specification

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Solution: $e \rightarrow a \wedge (b \vee (p \wedge r))$

Logic puzzles

- Puzzles that can be solved using logical reasoning are known as **logic puzzles**. Last class, we saw some examples. (Recall: Exactly j statements are false, j= 1, 2, …)

- Solving logic puzzles is an excellent way to practice working with the rules of logic.

- Computer programs designed to carry out logical reasoning often use well-known logic puzzles to illustrate their capabilities.

- We begin with a puzzle originally posed by Raymond Smullyan, a master of logic puzzles.

Logic puzzles

In [Sm78] Smullyan posed many puzzles about an island that has two kinds of inhabitants, knights, who always tell the truth, and their opposites, knaves, who always lie. You encounter two people A and B. What are A and B if A says "B is a knight" and B says "The two of us are opposite types?"

Solution: Let p and q be the statements that A is a knight and B is a knight, respectively, so that $\neg p$ and $\neg q$ are the statements that A is a knave and B is a knave, respectively.

We first consider the possibility that A is a knight; this is the statement that p is true. If A is a knight, then he is telling the truth when he says that B is a knight, so that q is true, and A and B are the same type. However, if B is a knight, then B's statement that A and B are of opposite types, the statement $(p \wedge \neg q) \vee (\neg p \wedge q)$, would have to be true, which it is not, because A and B are both knights. Consequently, we can conclude that A is not a knight, that is, that p is false.

If A is a knave, then because everything a knave says is false, A's statement that B is a knight, that is, that q is true, is a lie. This means that q is false and B is also a knave. Furthermore, if B is a knave, then B's statement that A and B are opposite types is a lie, which is consistent with both A and B being knaves. We can conclude that both A and B are knaves. ◀

Logic puzzles

Exercises 19–23 relate to inhabitants of the island of knights and knaves created by Smullyan, where knights always tell the truth and knaves always lie. You encounter two people, A and B. Determine, if possible, what A and B are if they address you in the ways described. If you cannot determine what these two people are, can you draw any conclusions?

19. A says "At least one of us is a knave" and B says nothing.

20. A says "The two of us are both knights" and B says "A is a knave."

21. A says "I am a knave or B is a knight" and B says nothing.

22. Both A and B say "I am a knight."

23. A says "We are both knaves" and B says nothing.

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22. Both *A* and *B* say "I am a knight."

23. *A* says "We are both knaves" and *B* says nothing.

Solution: (19) Suppose A is a knight. Then, A is telling the truth and hence one of them is a knave. Since A is a knight, B has to be a knave. The other alternative is not possible since a knave can't truthfully state that A or B is a knave.

Conclusion: A is a knight, B is a knave.

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21. *A* says "I am a knave or *B* is a knight" and *B* says nothing.

22. Both *A* and *B* say "I am a knight."

23. *A* says "We are both knaves" and *B* says nothing.

Solution: (21) Suppose A is a knave. Then what A says is false. Thus, A is a knight and B is a knave. But this is not possible, so A is a knight and hence the statement made by A is true. This means B is a knight.

Conclusion: Both are knights.

PROBLEMS 20 and 23 are assigned class work problems.

Logic puzzles

Exercise 35, page 24

A detective has interviewed four witnesses to a crime. From the stories of the witnesses the detective has concluded that if the butler is telling the truth then so is the cook; the cook and the gardener cannot both be telling the truth; the gardener and the handyman are not both lying; and if the handyman is telling the truth then the cook is lying. For each of the four witnesses, can the detective determine whether that person is telling the truth or lying? Explain your reasoning.

We will solve this problem in class.

Logic puzzles

Exercise 35, page 24

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Answer: B and C are lying. Can't determine the other two.

Logic circuits

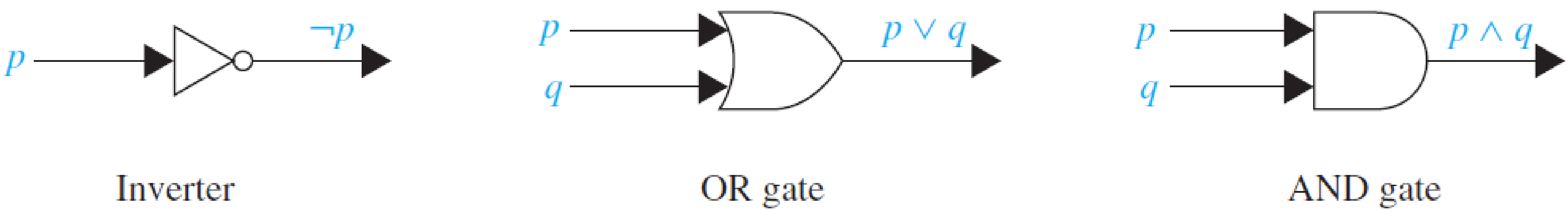
- In last lecture, we discussed how to create compound statements that combine propositions using operations: ~, V, ^, → etc.

- Computer hardware is based on the same framework. Inputs and outputs use binary signals 0, and 1.

- Circuits can be built to evaluate whether a compound expression (propositional formula) is true.

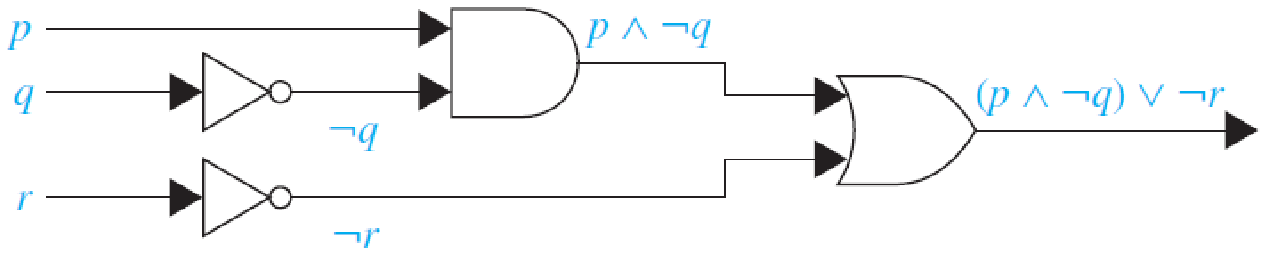
- The idea of building circuits using Boolean algebra as the basis was developed by Claude Shannon, who is also famous for his work on information theory which plays a central role in electronic communication.

Logic circuits



- Initially these building blocks were built using vacuum tubes, but since 50's, they are built using silicon.
- Computers are built using combinational logic together with memory to hold intermediate results of computation. (Such memory units are called volatile memory or RAM.)

A combinational circuit



Function computed by the circuit:

p	q	r	$(p \wedge \neg q)$	$(p \wedge \neg q) \vee \neg r$
true	true	true	false	false
true	false	true	true	true
false	true	true	false	false
false	false	true	false	false
true	true	false	false	true
true	false	false	true	true
false	true	false	false	true
false	false	false	false	true

Combinational Circuits

Design a circuit (using ~, ^ and V gates) that implements the following Boolean function F of three variables p, q and r.

p	q	r	F(p, q, r)
true	true	true	true
true	false	true	true
false	true	true	true
false	false	true	false
true	true	false	true
true	false	false	false
false	true	false	false
false	false	false	false

This Boolean function is called the majority function that takes value true if and only if at least two of its three inputs are true.

SOLUTION WILL BE PRESENTED IN CLASS.

Two important definitions

- A *tautology* is a proposition that's always TRUE.
- A *contradiction* is a proposition that's always FALSE.

Examples:

- $p \wedge \sim p$ (contradiction)
- $(p \rightarrow q) \vee (\sim q)$ (tautology)

Propositional Logic – a proof

- Show that $[p \wedge (p \rightarrow q)] \rightarrow q$ is a tautology.

Proof: We can use truth-table approach as we have done in previous examples.

- Another approach (often more efficient)
  - Deduction system: use rules to replace one propositional expression by an equivalent one, and repeat the process until the expression simplifies to T (which means the given expression is always True, i.e., it is a tautology.)

Propositional logic - Rules of equivalence

- There are many rules of equivalence that we can use.
- An important equivalence is

$$p \rightarrow q \equiv (\sim p \vee q)$$

TABLE 6 **Logical Equivalences.**

Equivalence	*Name*
$p \wedge \mathbf{T} \equiv p$ $p \vee \mathbf{F} \equiv p$	Identity laws
$p \vee \mathbf{T} \equiv \mathbf{T}$ $p \wedge \mathbf{F} \equiv \mathbf{F}$	Domination laws
$p \vee p \equiv p$ $p \wedge p \equiv p$	Idempotent laws
$\neg(\neg p) \equiv p$	Double negation law
$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutative laws
$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative laws
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive laws
$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$	De Morgan's laws
$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	Absorption laws
$p \vee \neg p \equiv \mathbf{T}$ $p \wedge \neg p \equiv \mathbf{F}$	Negation laws

- Each of these can be proved to be a tautology and hence we can replace the left-side of any of these by the corresponding right-side.

Propositional Logic – a proof

- Show that $[p \wedge (p \to q)] \to q$ is a tautology.

Proof: We use $\equiv$ to show that $[p \wedge (p \to q)] \to q \equiv T$.

$[p \wedge (p \to q)] \to q$	
$\equiv [p \wedge (\neg p \vee q)] \to q$	substitution for $\to$
$\equiv [(p \wedge \neg p) \vee (p \wedge q)] \to q$	distributive law
$\equiv [F \vee (p \wedge q)] \to q$	replacement
$\equiv (p \wedge q) \to q$	identity
$\equiv \neg(p \wedge q) \vee q$	substitution for $\to$
$\equiv (\neg p \vee \neg q) \vee q$	DeMorgan's law
$\equiv \neg p \vee (\neg q \vee q)$	associative
$\equiv \neg p \vee T$	excluded middle
$\equiv T$	domination

Propositional logic proof

Show that $\neg(p \vee (\neg p \wedge q))$ and $\neg p \wedge \neg q$ are logically equivalent by developing a series of logical equivalences.

Solution: We will use one of the equivalences in Table 6 at a time, starting with $\neg(p \vee (\neg p \wedge q))$ and ending with $\neg p \wedge \neg q$. (*Note:* we could also easily establish this equivalence using a truth table.) We have the following equivalences.

$$
\begin{aligned}
\neg(p \vee (\neg p \wedge q)) &\equiv \neg p \wedge \neg(\neg p \wedge q) && \text{by the second De Morgan law} \\
&\equiv \neg p \wedge [\neg(\neg p) \vee \neg q] && \text{by the first De Morgan law} \\
&\equiv \neg p \wedge (p \vee \neg q) && \text{by the double negation law} \\
&\equiv (\neg p \wedge p) \vee (\neg p \wedge \neg q) && \text{by the second distributive law} \\
&\equiv \mathbf{F} \vee (\neg p \wedge \neg q) && \text{because } \neg p \wedge p \equiv \mathbf{F} \\
&\equiv (\neg p \wedge \neg q) \vee \mathbf{F} && \text{by the commutative law for disjunction} \\
&\equiv \neg p \wedge \neg q && \text{by the identity law for } \mathbf{F}
\end{aligned}
$$

Consequently $\neg(p \vee (\neg p \wedge q))$ and $\neg p \wedge \neg q$ are logically equivalent. ◀

Exercise, Section 1.3

The following exercise involves the logical operators *NAND* and *NOR*. The proposition *p NAND q* is true when either *p* or *q*, or both, are false; and it is false when both *p* and *q* are true. The proposition *p NOR q* is true when both *p* and *q* are false, and it is false otherwise. The proposition *p NOR q* are denoted by $p \mid q$ and $p \downarrow q$, respectively. (The operators | and ↓ are called the **Sheffer stroke** and the **Peirce arrow** after H. *M.* Sheffer and C. *S.* Peirce, respectively.

Exercise 51: Find a compound proposition logically equivalent to $p \rightarrow q$ using only the logical operator ↓.

Exercise, Section 1.3

Exercise 55: How many different truth tables of compound propositions are there that involve the propositional variables p and q?

Predicate Logic

Alicia eats pizza at least once a week.
Garrett eats pizza at least once a week.
Allison eats pizza at least once a week.
Gregg eats pizza at least once a week.
Ryan eats pizza at least once a week.
Meera eats pizza at least once a week.
Ariel eats pizza at least once a week.

⋮

Predicates

Alicia eats pizza at least once a week.

⋮

Define:

EP(x) = "x eats pizza at least once a week."

A *predicate*, or propositional function, is a function that takes some variable(s) as arguments and returns True or False.

Note that *EP(x)* is not a proposition, *EP(*Ariel*)* is.

Predicates

A predicate is a function of some variables (for example x, y, …).

The value taken by variable belongs to some set called its domain,

and the value taken by the function will always be true or false.

Example: taking242 is a predicate with domain D = set of all CS majors. For a specific student, e.g., Tom, taking242(Tom) will be true if and only if Tom is currently taking242.

Predicates

Suppose $Q(x,y)$ denotes the predicate "$x > y$"

True or False ?

$Q(4, 3)$

$Q(3, 4)$

$Q(3, 9) \lor Q(9, 3)$

Predicates - the universal quantifier

Another way of changing a predicate into a proposition.

Suppose P(x) is a predicate on some universe of discourse.

Ex. B(x) = "x is carrying a backpack," x is set of cs242 students.

The universal quantifier of P(x) is the **statement**:

"P(x) is true for all x in the universe of discourse."

We write it $\forall x\, P(x)$, and say "for all x, P(x)"

$\forall x\, P(x)$ is TRUE if P(x) is true for every single x.

$\forall x\, P(x)$ is FALSE if there is an x for which P(x) is false.

Predicates - the universal quantifier

B(x) = "x is wearing sneakers."
L(x) = "x is at least 21 years old."
Y(x)= "x is less than 24 years old."

Are either of these propositions true?

a) $\forall x\ (Y(x) \rightarrow B(x))$
b) $\forall x\ (Y(x) \vee L(x))$

Predicates - the existential quantifier

Another way of changing a predicate into a proposition.

Suppose P(x) is a predicate on some universe of discourse.

Ex. C(x) = "x has a candy bar," x is set of cs242 students.

The existential quantifier of P(x) is the **proposition**:

"P(x) is true for some x in the universe of discourse."

We write it ∃x P(x), and say "for some x, P(x)"

∃x P(x) is TRUE if there is an x for which P(x) is true.

∃x P(x) is FALSE if P(x) is false for every single x.

Predicates - the existential quantifier

B(x) = "x is majoring in computer science."
L(x) = "x is at least 21 years old."
Y(x)= "x is less than 24 years old."

Universe of discourse is people in this room.

Which of these propositions true?

a) $\exists x\, B(x)$
b) $\exists x\, (Y(x) \wedge L(x))$
c) $\exists x\, (Y(x)) \wedge \exists x\, (L(x))$

Predicates - more examples

$L(x)$ = "x is a lion."
$F(x)$ = "x is fierce."
$C(x)$ = "x drinks coffee."

Universe of discourse is all creatures.

All lions are fierce.

$\forall x\,(L(x) \rightarrow F(x))$

Some lions don't drink coffee.

$\exists x\,(L(x) \wedge \neg C(x))$

Some fierce creatures don't drink coffee.

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Some lions don’t drink coffee.

Some fierce creatures don’t drink coffee.

$\exists x\,(F(x) \wedge \neg C(x))$

Predicates - more examples

B(x) = "x is a hummingbird."
L(x) = "x is a large bird."
H(x) = "x lives on honey."
R(x) = "x is richly colored."

Universe of discourse is all creatures.

All hummingbirds are richly colored.

No large birds live on honey.

Birds that do not live on honey are dully colored.

Predicates - more examples

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L(x) = “x is a large bird.”
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Universe of discourse is all creatures.

All hummingbirds are richly colored.

$\forall x\ (B(x) \rightarrow R(x))$

No large birds live on honey.

Birds that do not live on honey are dully colored.

Predicates - quantifier negation

No large birds live on honey.

$\neg\exists x\,(L(x) \wedge H(x))$

$\exists x\, P(x)$ means "P(x) is true for some x."

What about $\neg\exists x\, P(x)$?

Not ["P(x) is true for some x."]

"P(x) is not true for all x."

$\forall x\, \neg P(x)$

So, $\neg\exists x\, P(x)$ is the same as $\forall x\, \neg P(x)$.

$\forall x\, \neg(L(x) \wedge H(x))$

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Universe of discourse is all creatures.

All hummingbirds are richly colored.

No large birds live on honey. $\neg\exists x\,(L(x) \wedge H(x))$

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No large birds live on honey.

Birds that do not live on honey are dully colored.

$\forall x\ (\neg H(x) \rightarrow \neg R(x))$

Predicates - quantifier negation

Not all large birds live on honey.

$\neg\forall x\,(L(x) \rightarrow H(x))$

$\forall x\, P(x)$ means "$P(x)$ is true for every x."

What about $\neg\forall x\, P(x)$?

Not ["$P(x)$ is true for every x."]

"There is an x for which $P(x)$ is not true."

$\exists x\, \neg P(x)$

So, $\neg\forall x\, P(x)$ is the same as $\exists x\, \neg P(x)$.

$\exists x\, \neg(L(x) \rightarrow H(x))$

Predicates - quantifier negation

So, $\neg\forall x\, P(x)$ is the same as $\exists x\, \neg P(x)$.
So, $\neg\exists x\, P(x)$ is the same as $\forall x\, \neg P(x)$.

General rule: to negate a quantifier, move negation to the right, changing quantifiers as you go.