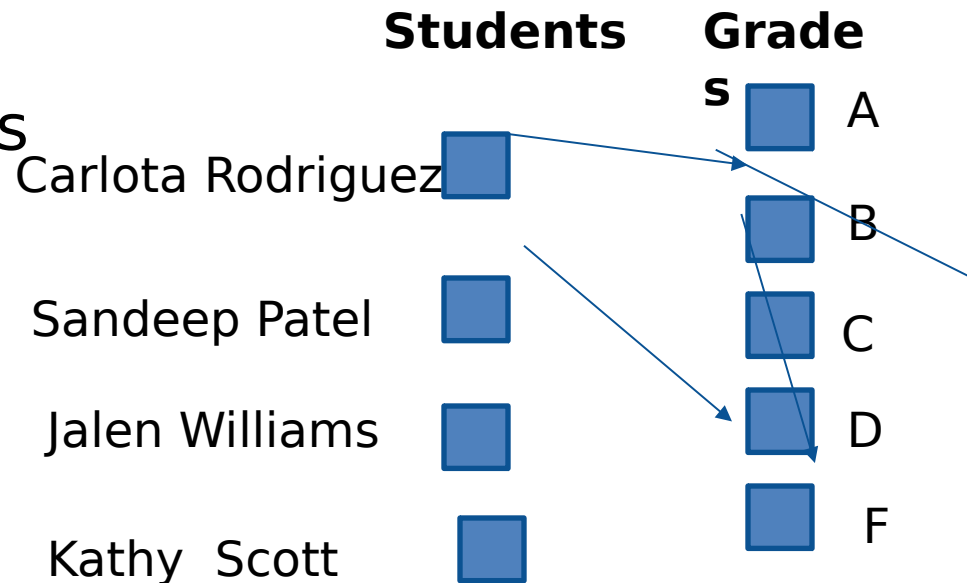


Functions

Definition: Let A and B be nonempty sets. A *function* f from A to B , denoted $f: A \rightarrow B$ is an assignment of each element of A to exactly one element of B . We write $f(a) = b$ if b is the unique element of B assigned by the function f to the element a of A .

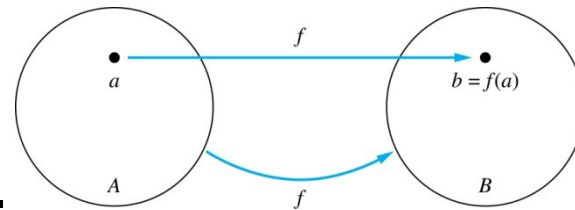
- Functions are sometimes called *mappings* or *transformations*.



Functions

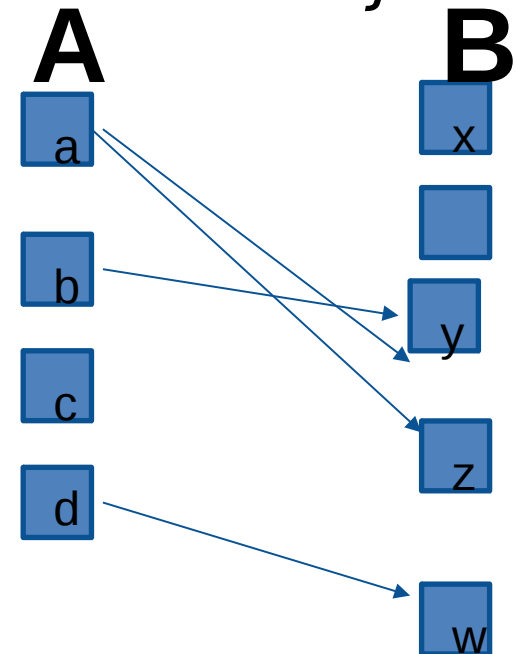
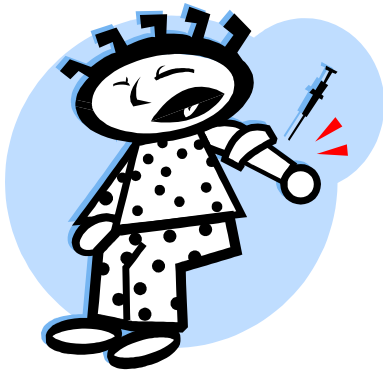
Given a function $f: A \rightarrow B$:

- We say f maps A to B or f is a *mapping* from A to B .
- A is called the *domain* of f .
- B is called the *codomain* of f .
- If $f(a) = b$,
 - then b is called the *image* of a under f .
 - a is called the *preimage* of b .
- The range of f is the set of all images of points in **A** under f . We denote it by $f(A)$.
- Two functions are *equal* when they have the same domain, the same codomain and map each element of the domain to the same element of the codomain.



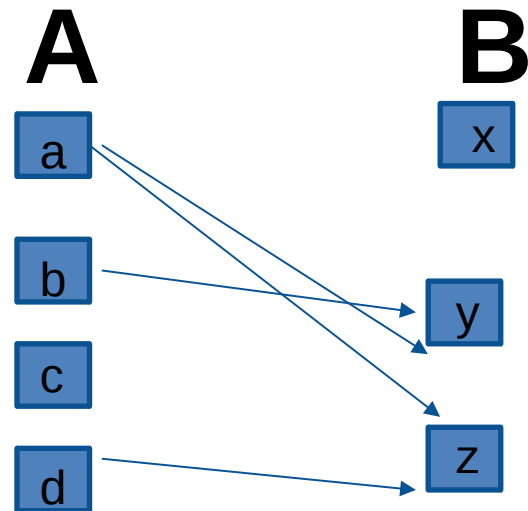
Injectons

Definition: A function f is said to be *one-to-one*, or *injective*, if and only if $f(a) = f(b)$ implies that $a = b$ for all a and b in the domain of f . A function is said to be an *injection* if it is one-to-one.



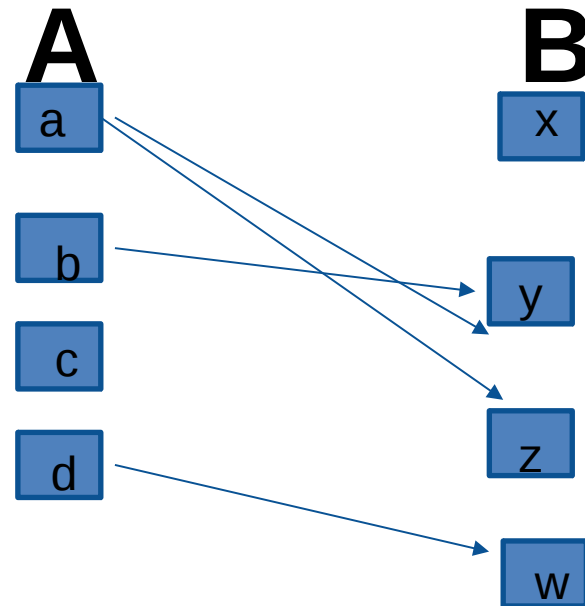
Surjections

Definition: A function f from A to B is called *onto* or *surjective*, if and only if for every element $a \in A$ there is an element $b \in B$ with $f(a) = b$. A function f is called a *surjection* if it is onto.



Bijections

Definition: A function f is a *one-to-one correspondence*, or a *bijection*, if it is both one-to-one and onto (surjective and injective).



Showing that f is one-to-one or onto

Suppose that $f : A \rightarrow B$.

To show that f is injective Show that if $f(x) = f(y)$ for arbitrary $x, y \in A$ with $x \neq y$, then $x = y$.

To show that f is not injective Find particular elements $x, y \in A$ such that $x \neq y$ and $f(x) = f(y)$.

To show that f is surjective Consider an arbitrary element $y \in B$ and find an element $x \in A$ such that $f(x) = y$.

To show that f is not surjective Find a particular $y \in B$ such that $f(x) \neq y$ for all $x \in A$.

Showing that f is one-to-one or onto

Example 1: Let f be the function from $\{a,b,c,d\}$ to $\{1,2,3\}$ defined by $f(a) = 3$, $f(b) = 2$, $f(c) = 1$, and $f(d) = 3$. Is f an onto function?

Solution: Yes, f is onto since all three elements of the codomain are images of elements in the domain. If the codomain were changed to $\{1,2,3,4\}$, f would not be onto.

Example 2: Is the function $f(x) = x^2$ from the set of integers onto?

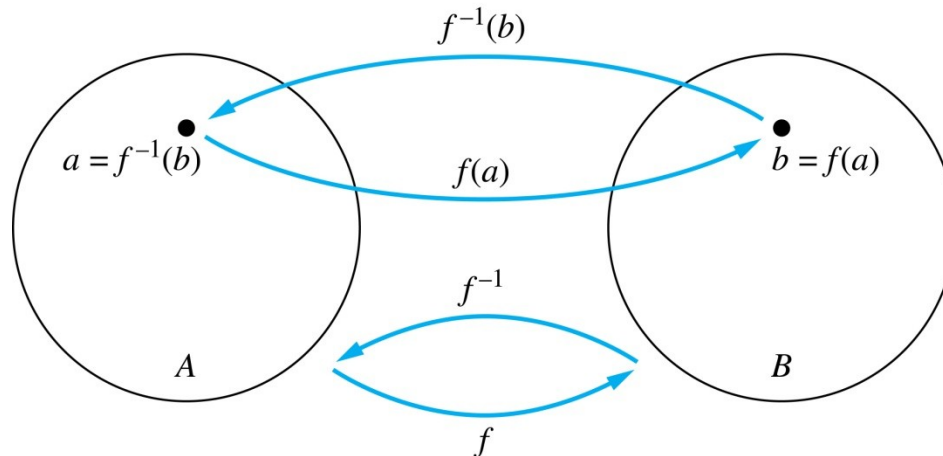
Solution: No, f is not onto because there is no integer x with $x^2 = -1$, for example.

Inverse Functions

Definition: Let f be a bijection from A to B . Then the *inverse* of f , denoted f^{-1} , is the function from B to A defined as

$$f^{-1}(y) = x \text{ iff } f(x) = y$$

No inverse exists unless f is a bijection. Why?



Questions

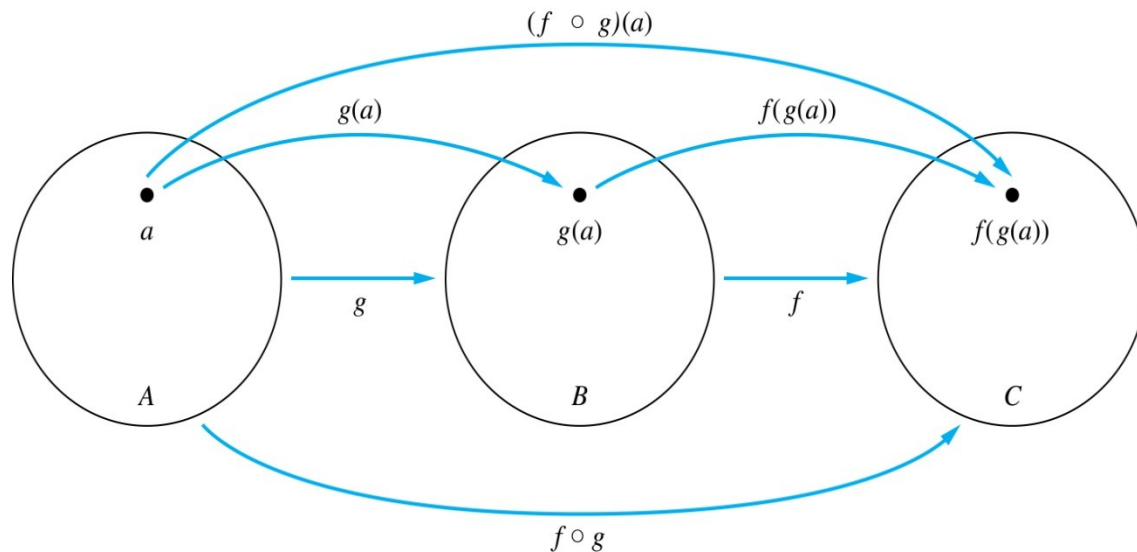
Example 1: Let f be the function from $\{a,b,c\}$ to $\{1,2,3\}$ such that $f(a) = 2$, $f(b) = 3$, and $f(c) = 1$. Is f invertible and if so what is its inverse?

Solution: The function f is invertible because it is a one-to-one correspondence. The inverse function f^{-1} reverses the correspondence given by f , so $f^{-1}(1) = c$, $f^{-1}(2) = a$, and $f^{-1}(3) = b$.

Composition

- **Definition:** Let $f: B \rightarrow C$, $g: A \rightarrow B$. The *composition of f with g* , denoted $f \circ g$ is the function from A to C defined by

$$f \circ g(x) = f(g(x))$$



Composition

Example 1: If $f(x) = x^2$ and $g(x) = 2x + 1$, then

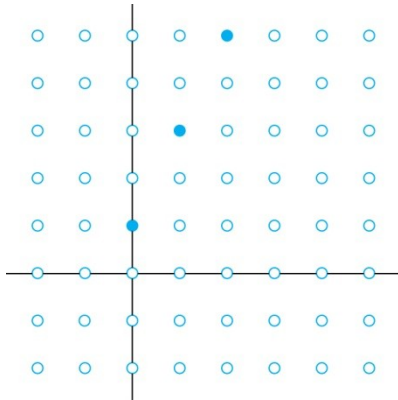
$$f(g(x)) = (2x + 1)^2$$

and

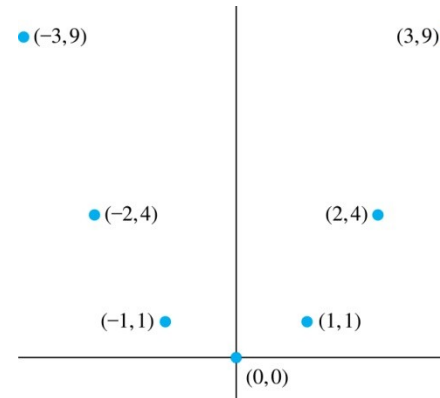
$$g(f(x)) = 2x^2 + 1$$

Graphs of Functions

- Let f be a function from the set A to the set B . The *graph* of the function f is the set of ordered pairs $\{(a,b) \mid a \in A \text{ and } f(a) = b\}$.



Graph of $f(n) = 2n + 1$
from \mathbb{Z} to \mathbb{Z}



Graph of $f(x) = x^2$
from \mathbb{Z} to \mathbb{Z}

Proving Properties of Functions

Example: Prove that x is a real number, then

$$\lfloor 2x \rfloor = \lfloor x \rfloor + \lfloor x + 1/2 \rfloor$$

Solution: Let $x = n + \varepsilon$, where n is an integer and $0 \leq \varepsilon < 1$.

Case 1: $\varepsilon < 1/2$

- $2x = 2n + 2\varepsilon$ and $\lfloor 2x \rfloor = 2n$, since $0 \leq 2\varepsilon < 1$.
- $\lfloor x + 1/2 \rfloor = n$, since $x + 1/2 = n + (1/2 + \varepsilon)$ and $0 \leq 1/2 + \varepsilon < 1$.
- Hence, $\lfloor 2x \rfloor = 2n$ and $\lfloor x \rfloor + \lfloor x + 1/2 \rfloor = n + n = 2n$.

Case 2: $\varepsilon \geq 1/2$

- $2x = 2n + 2\varepsilon = (2n + 1) + (2\varepsilon - 1)$ and $\lfloor 2x \rfloor = 2n + 1$,
since $0 \leq 2\varepsilon - 1 < 1$.
- $\lfloor x + 1/2 \rfloor = \lfloor n + (1/2 + \varepsilon) \rfloor = \lfloor n + 1 + (\varepsilon - 1/2) \rfloor = n + 1$ since
 $0 \leq \varepsilon - 1/2 < 1$.
- Hence, $\lfloor 2x \rfloor = 2n + 1$ and $\lfloor x \rfloor + \lfloor x + 1/2 \rfloor = n + (n + 1) = 2n + 1$. ■

Sequences

Definition: A *sequence* is a function from a subset of the integers (usually either the set $\{0, 1, 2, 3, 4, \dots\}$ or $\{1, 2, 3, 4, \dots\}$) to a set S .

- The notation a_n is used to denote the image of the integer n . We can think of a_n as the equivalent of $f(n)$ where f is a function from $\{0, 1, 2, \dots\}$ to S . We call a_n a *term* of the sequence.

Sequences

Example: Consider the sequence $\{a_n\}$ where

$$a_n = \frac{1}{n} \quad \{a_n\} = \{a_1, a_2, a_3, \dots\}$$

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4} \dots$$

Geometric Progression

Definition: A *geometric progression* is a sequence of the form: $a, ar, ar^2, \dots, ar^n, \dots$
where the *initial term* a and the *common ratio* r are real numbers.

Examples:

1. Let $a = 1$ and $r = -1$. Then:

$$\{b_n\} = \{b_0, b_1, b_2, b_3, b_4, \dots\} = \{1, -1, 1, -1, 1, \dots\}$$

2. Let $a = 2$ and $r = 5$. Then:

$$\{c_n\} = \{c_0, c_1, c_2, c_3, c_4, \dots\} = \{2, 10, 50, 250, 1250, \dots\}$$

3. Let $a = 6$ and $r = 1/3$. Then:

$$\{d_n\} = \{d_0, d_1, d_2, d_3, d_4, \dots\} = \{6, 2, \frac{2}{3}, \frac{2}{9}, \frac{2}{27}, \dots\}$$

Arithmetic Progression

Definition: A *arithmetic progression* is a sequence of the form: $a, a + d, a + 2d, \dots, a + nd, \dots$

where the *initial term* a and the *common difference* d are real numbers.

Examples:

1. Let $a = -1$ and $d = 4$:

$$\{s_n\} = \{s_0, s_1, s_2, s_3, s_4, \dots\} = \{-1, 3, 7, 11, 15, \dots\}$$

2. Let $a = 7$ and $d = -3$:

$$\{t_n\} = \{t_0, t_1, t_2, t_3, t_4, \dots\} = \{7, 4, 1, -2, -5, \dots\}$$

3. Let $a = 1$ and $d = 2$:

$$\{u_n\} = \{u_0, u_1, u_2, u_3, u_4, \dots\} = \{1, 3, 5, 7, 9, \dots\}$$

Strings

Definition: A *string* is a finite sequence of characters from a finite set (an alphabet).

- Sequences of characters or bits are important in computer science.
- The *empty string* is represented by λ .
- The string *abcde* has *length* 5.

Recurrence Relations

Definition: A *recurrence relation* for the sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more of the previous terms of the sequence, namely, a_0, a_1, \dots, a_{n-1} , for all integers n with $n \geq n_0$, where n_0 is a nonnegative integer.

- A sequence is called a *solution* of a recurrence relation if its terms satisfy the recurrence relation.
- The *initial conditions* for a sequence specify the terms that precede the first term where the recurrence relation takes effect.

Questions about Recurrence Relations

Example 1: Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} + 3$ for $n = 1, 2, 3, 4, \dots$ and suppose that $a_0 = 2$. What are a_1 , a_2 and a_3 ?

[Here $a_0 = 2$ is the initial condition.]

Solution: We see from the recurrence relation that

$$a_1 = a_0 + 3 = 2 + 3 = 5$$

$$a_2 = 5 + 3 = 8$$

$$a_3 = 8 + 3 = 11$$

Solving Recurrence Relations

- Finding a formula for the n th term of the sequence generated by a recurrence relation is called *solving the recurrence relation*.
- Such a formula is called a *closed formula*.

Iterative Solution

Example

Method 1: Working upward, forward substitution

Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} + 3$ for $n = 2, 3, 4, \dots$ and suppose that $a_1 = 2$.

$$a_2 = 2 + 3$$

$$a_3 = (2 + 3) + 3 = 2 + 3 \cdot 2$$

$$a_4 = (2 + 2 \cdot 3) + 3 = 2 + 3 \cdot 3$$

.

.

.

$$a_n = a_{n-1} + 3 = (2 + 3 \cdot (n - 2)) + 3 = 2 + 3(n - 1)$$

Iterative Solution Example

Method 2: Working downward, backward substitution

Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} + 3$ for $n = 2, 3, 4, \dots$ and suppose that $a_1 = 2$.

$$\begin{aligned} a_n &= a_{n-1} + 3 \\ &= (a_{n-2} + 3) + 3 = a_{n-2} + 3 \cdot 2 \\ &= (a_{n-3} + 3) + 3 \cdot 2 = a_{n-3} + 3 \cdot 3 \\ &\vdots \\ &\vdots \\ &\vdots \end{aligned}$$

Financial Application

Example: Suppose that a person deposits \$10,000.00 in a savings account at a bank yielding 11% per year with interest compounded annually. How much will be in the account after 30 years?

Let P_n denote the amount in the account after n years. P_n satisfies the following recurrence relation:

$$P_n = P_{n-1} + 0.11P_{n-1} = (1.11) P_{n-1}$$

with the initial condition $P_0 = 10,000$

Continued on next slide □

Financial Application

Solution: Forward Substitution

$$P_1 = (1.11)P_0$$

$$P_2 = (1.11)P_1 = (1.11)^2P_0$$

$$P_3 = (1.11)P_2 = (1.11)^3P_0$$

⋮

$$P_n = (1.11)P_{n-1} = (1.11)^nP_0 = (1.11)^n 10,000$$

$P_n = (1.11)^n 10,000$ (Can prove by induction, covered in Chapter 5)

$$P_{30} = (1.11)^{30} 10,000 = \$228,992.97$$

Useful Sequences

TABLE 1 Some Useful Sequences.

<i>n</i> th Term	First 10 Terms
n^2	1, 4, 9, 16, 25, 36, 49, 64, 81, 100, ...
n^3	1, 8, 27, 64, 125, 216, 343, 512, 729, 1000, ...
n^4	1, 16, 81, 256, 625, 1296, 2401, 4096, 6561, 10000, ...
2^n	2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, ...
3^n	3, 9, 27, 81, 243, 729, 2187, 6561, 19683, 59049, ...
$n!$	1, 2, 6, 24, 120, 720, 5040, 40320, 362880, 3628800, ...
f_n	1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, ...

Summations

- Sum of the terms a_m, a_{m+1}, \dots, a_n from the sequence $\{a_n\}$
- The notation:

represents
$$\sum_{j=m}^n a_j \quad \sum_{j=m}^n a_j \quad \sum_{m \leq j \leq n} a_j$$

$$a_m + a_{m+1} + \dots + a_n$$

- The variable j is called the *index of summation*. It runs through all the integers starting with its *lower limit* m and ending with its *upper limit* n .

Product Notation (*optional*)

- Product of the terms a_m, a_{m+1}, \dots, a_n from the sequence $\{a_n\}$
- The notation:

$$\prod_{j=m}^n a_j \quad \prod_{j=m}^n a_j \quad \prod_{m \leq j \leq n} a_j$$

represents

$$a_m \times a_{m+1} \times \cdots \times a_n$$

Geometric Series

Sums of terms of geometric progressions

$$\sum_{j=0}^n ar^j = \begin{cases} \frac{ar^{n+1} - a}{r - 1} & r \neq 1 \\ (n + 1)a & r = 1 \end{cases}$$

Proof: Let $S_n = \sum_{j=0}^n ar^j$ To compute S_n , first multiply both sides of the equality by r and then manipulate the resulting sum as follows:

$$\begin{aligned} rS_n &= r \sum_{j=0}^n ar^j \\ &= \sum_{j=0}^n ar^{j+1} \end{aligned}$$

Continued on next slide \square

Geometric Series

$$= \sum_{j=0}^n ar^{j+1} \quad \text{From previous slide.}$$

$$= \sum_{k=1}^{n+1} ar^k \quad \text{Shifting the index of summation with } k = j + 1.$$

$$= \left(\sum_{k=0}^n ar^k \right) + (ar^{n+1} - a) \quad \text{Removing } k = n + 1 \text{ term and adding } k = 0 \text{ term.}$$

$$= S_n + (ar^{n+1} - a) \quad \text{Substituting } S \text{ for summation formula}$$

∴

$$rS_n = S_n + (ar^{n+1} - a)$$

$$S_n = \frac{ar^{n+1} - a}{r - 1} \quad \text{if } r \neq 1$$

$$S_n = \sum_{j=0}^n ar^j = \sum_{j=0}^n a = (n + 1)a \quad \text{if } r = 1$$

Some Useful Summation Formulae

TABLE 2 Some Useful Summation Formulae.	
<i>Sum</i>	<i>Closed Form</i>
$\sum_{k=0}^n ar^k \ (r \neq 0)$	$\frac{ar^{n+1} - a}{r - 1}, r \neq 1$
$\sum_{k=1}^n k$	$\frac{n(n+1)}{2}$
$\sum_{k=1}^n k^2$	$\frac{n(n+1)(2n+1)}{6}$
$\sum_{k=1}^n k^3$	$\frac{n^2(n+1)^2}{4}$
$\sum_{k=0}^{\infty} x^k, x < 1$	$\frac{1}{1-x}$
$\sum_{k=1}^{\infty} kx^{k-1}, x < 1$	$\frac{1}{(1-x)^2}$

Geometric Series:
We just proved this.

Later we will prove some of these by induction.

Proof in text (requires calculus)

Cardinality

Definition: The *cardinality* of a set A is equal to the cardinality of a set B , denoted

$$|A| = |B|,$$

if and only if there is a one-to-one correspondence (*i.e.*, a bijection) from A to B .

- If there is a one-to-one function (*i.e.*, an injection) from A to B , the cardinality of A is less than or the same as the cardinality of B and we write $|A| \leq |B|$.
- When $|A| \leq |B|$ and A and B have different cardinality, we say that the cardinality of A is less than the cardinality of B and write $|A| < |B|$.

Cardinality

- **Definition:** A set that is either finite or has the same cardinality as the set of positive integers (\mathbf{Z}^+) is called *countable*. A set that is not countable is *uncountable*.
- The set of real numbers \mathbf{R} is an uncountable set.
- When an infinite set is countable (*countably infinite*) its cardinality is \aleph_0 (where \aleph is aleph, the 1st letter of the Hebrew alphabet). We write $|S| = \aleph_0$ and say that S has cardinality “aleph null.”

Showing that a Set is Countable

- An infinite set is countable if and only if it is possible to list the elements of the set in a sequence (indexed by the positive integers).
- The reason for this is that a one-to-one correspondence f from the set of positive integers to a set S can be expressed in terms of a sequence $a_1, a_2, \dots, a_n, \dots$ where $a_1 = f(1)$, $a_2 = f(2)$, \dots , $a_n = f(n)$, \dots

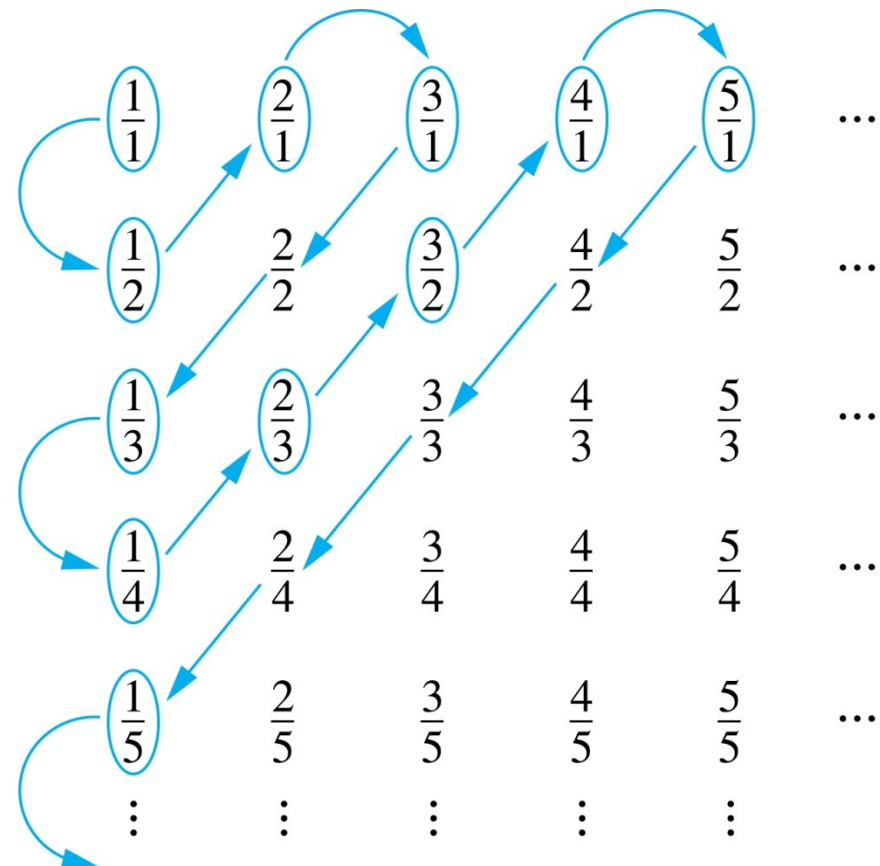
The Positive Rational Numbers are Countable

First row $q = 1$.
 Second row $q = 2$.
 etc.

Constructing the List

First list p/q with $p + q = 2$.
 Next list p/q with $p + q = 3$.
 And so on.

1, $\frac{1}{2}$, 2, 3, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{2}{3}$,



The Positive Rational Numbers are Countable

- **Definition:** A *rational number* can be expressed as the ratio of two integers p and q such that $q \neq 0$.
 - $\frac{3}{4}$ is a rational number
 - $\sqrt{2}$ is not a rational number.

Example 1: Show that the positive rational numbers are countable.

Solution: The positive rational numbers are countable since they can be arranged in a sequence:

$$r_1, r_2, r_3, \dots$$

The next slide shows how this is done.

Strings

Example 2: Show that the set of finite strings S over a finite alphabet A is countably infinite.

Assume an alphabetical ordering of symbols in A

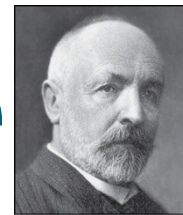
Solution: Show that the strings can be listed in a sequence. First list

1. All the strings of length 0 in alphabetical order.
2. Then all the strings of length 1 in lexicographic (as in a dictionary) order.
3. Then all the strings of length 2 in lexicographic order.
4. And so on. ■

This implies a bijection from \mathbf{N} to S and hence it is a countably infinite set.

The Real Numbers are Uncountable

Georg Cantor
(1845-1918)



Example3: Show that the set of real numbers is uncountable.

Solution: The method is called the Cantor diagonalization argument, and is a proof by contradiction.

$$\begin{aligned} r_1 &= 0.d_{11}d_{12}d_{13}d_{14}d_{15}d_{16}\dots \\ r_2 &= 0.d_{21}d_{22}d_{23}d_{24}d_{25}d_{26}\dots \\ r_3 &= 0.d_{31}d_{32}d_{33}d_{34}d_{35}d_{36}\dots \\ &\vdots \end{aligned}$$

1. Suppose \mathbf{R} is countable. Then the real numbers between 0 and 1 are also countable (any subset of a countable set is countable - an exercise in the text).
2. The real numbers between 0 and 1 can be listed in order r_1, r_2, r_3, \dots .



Matrix

Definition: A *matrix* is a rectangular array of numbers. A matrix with m rows and n columns is called an $m \times n$ matrix.

- The plural of matrix is *matrices*.
- A matrix with the same number of rows as columns is called *square*.
- Two matrices are *equal* if they have the same number of rows and the same number of columns and the corresponding entries in every position are equal.

3×2 matrix

$$\begin{bmatrix} 1 & 1 \\ 0 & 2 \\ 1 & 3 \end{bmatrix}$$

Notation

- Let m and n be positive integers and let

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

- The i th row of \mathbf{A} is the $1 \times n$ matrix $[a_{i1}, a_{i2}, \dots, a_{in}]$. The j th column of \mathbf{A} is the $m \times 1$ matrix:

$$\begin{bmatrix} a_{1j} \\ a_{2j} \\ \cdot \\ \cdot \\ a_{mj} \end{bmatrix}$$

- The (i,j) th *element* or *entry* of \mathbf{A} is the element a_{ij} . We can use $\mathbf{A} = [a_{ij}]$ to denote the matrix with its (i,j) th element equal to a_{ij} .

Matrix Arithmetic: Addition

Definition: Let $\mathbf{A} = [a_{ij}]$ and $\mathbf{B} = [b_{ij}]$ be $m \times n$ matrices. The sum of \mathbf{A} and \mathbf{B} , denoted by $\mathbf{A} + \mathbf{B}$, is the $m \times n$ matrix that has $a_{ij} + b_{ij}$ as its (i,j) th element. In other words, $\mathbf{A} + \mathbf{B} = [a_{ij} + b_{ij}]$.

Example:

$$\begin{bmatrix} 1 & 0 & -1 \\ 2 & 2 & -3 \\ 3 & 4 & 0 \end{bmatrix} + \begin{bmatrix} 3 & 4 & -1 \\ 1 & -3 & 0 \\ -1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 4 & -2 \\ 3 & -1 & -3 \\ 2 & 5 & 2 \end{bmatrix}$$

Note that matrices of different sizes can not be added.

Matrix Multiplication

Definition: Let \mathbf{A} be an $n \times k$ matrix and \mathbf{B} be a $k \times n$ matrix. The *product* of \mathbf{A} and \mathbf{B} , denoted by \mathbf{AB} , is the $m \times n$ matrix that has its (i,j) th element equal to the sum of the products of the corresponding elements from the i th row of \mathbf{A} and the j th column of \mathbf{B} . In other words, if $\mathbf{AB} = [c_{ij}]$ then $c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \dots + a_{kj}b_{kj}$.

Example:

$$\begin{bmatrix} 1 & 0 & 4 \\ 2 & 1 & 1 \\ 3 & 1 & 0 \\ 0 & 2 & 2 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 1 & 1 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 14 & 4 \\ 8 & 9 \\ 7 & 13 \\ 8 & 2 \end{bmatrix}$$

The product of two matrices is undefined when the number of columns in the first matrix is not the same as the number of rows in the second.

Illustration of Matrix Multiplication

- The Product of $\mathbf{A} = [a_{ij}]$ and $\mathbf{B} = [b_{ij}]$

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1k} \\ a_{21} & a_{22} & \dots & a_{2k} \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ \textcolor{red}{a_{i1}} & \textcolor{red}{a_{i2}} & \dots & \textcolor{red}{a_{ik}} \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mk} \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} b_{11} & b_{12} & \dots & \textcolor{red}{b_{1j}} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & \textcolor{red}{b_{2j}} & \dots & b_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ \vdots & \vdots & & \vdots & & \vdots \\ b_{k1} & b_{k2} & \dots & \textcolor{red}{b_{kj}} & \dots & b_{kn} \end{bmatrix}$$
$$\mathbf{AB} = \begin{bmatrix} c_{11} & c_{12} & \dots & c_{1n} \\ c_{21} & c_{22} & \dots & c_{2n} \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & \textcolor{red}{c_{ij}} & \vdots \\ \vdots & \vdots & & \vdots \\ c_{m1} & c_{m2} & \dots & c_{mn} \end{bmatrix}$$

$$\textcolor{red}{c_{ij}} = a_{i1}b_{1j} + a_{i2}b_{2j} + \dots + a_{ik}b_{kj}$$

Matrix Multiplication is not Commutative

Example: Let

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

Does **$\mathbf{AB} = \mathbf{BA}$** ?

Solution:

$$\mathbf{AB} = \begin{bmatrix} 2 & 2 \\ 5 & 3 \end{bmatrix}$$

$$\mathbf{BA} = \begin{bmatrix} 4 & 3 \\ 3 & 2 \end{bmatrix}$$

$$\mathbf{AB} \neq \mathbf{BA}$$

Identity Matrix and Powers of Matrices

Definition: The *identity matrix* of order n is the $n \times n$ matrix $\mathbf{I}_n = [\delta_{ij}]$, where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$.

$$\mathbf{I}_n = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

$$= \mathbf{I}_m \mathbf{A} = \mathbf{A} \mathbf{I}_n$$

$$\mathbf{A} \mathbf{I}_n$$

\times

when \mathbf{A} is an $m \times n$ matrix

Powers of square matrices can be defined.

When \mathbf{A} is an $n \times n$ matrix, we have:

$$\mathbf{A}^0 = \mathbf{I}_n \quad \mathbf{A}^r = \underbrace{\mathbf{A} \mathbf{A} \dots \mathbf{A}}_{r \text{ times}}$$

Transposes of Matrices

Definition: Let $\mathbf{A} = [a_{ij}]$ be an $m \times n$ matrix. The *transpose* of \mathbf{A} , denoted by \mathbf{A}^t , is the $n \times m$ matrix obtained by interchanging the rows and columns of \mathbf{A} .

If $\mathbf{A}^t = [b_{ij}]$, then $b_{ij} = a_{ji}$ for $i = 1, 2, \dots, n$
and $j = 1, 2, \dots, m$.

The transpose of the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ is the matrix $\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$.

Transposes of Matrices

Definition: A square matrix **A** is called symmetric if $\mathbf{A} = \mathbf{A}^t$. Thus $\mathbf{A} = [a_{ij}]$ is symmetric if $a_{ij} = a_{ji}$ for i and j with $1 \leq i \leq n$ and $1 \leq j \leq n$.

The matrix $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ is square.

Square matrices do not change when their rows and columns are interchanged.