CS 242 Discrete Structures **Spring 2014**

HW 2 solutions

Section 1.5

10. **a)** $\forall x F(x, \text{Fred})$ **b)** $\forall y F(\text{Evelyn}, y)$ **c)** $\forall x \exists y F(x, y)$ **d)** $\neg \exists x \forall y F(x, y)$ **e)** $\forall y \exists x F(x, y)$

f) $\neg \exists x(F(x, \text{Fred}) \land F(x, \text{Jerry}))$

g) $\exists y_1 \exists y_2 (F(\text{Nancy}, y_1) \land F(\text{Nancy}, y_2) \land y_1 \neq y_2 \land \forall y(F(\text{Nancy}, y) \rightarrow (y = y_1 \lor y = y_2)))$

h) $\exists y(\forall x F(x, y) \land \forall z(\forall x F(x, z) \rightarrow z = y))$ **i)** $\neg \exists x F(x, x)$

j) $\exists x \exists y(x \neq y \land F(x, y) \land \forall z((F(x, z) \land z \neq x) \rightarrow z = y))$ (We do not assume that this sentence is asserting that this person can or cannot fool her/himself.)

16. We let $P(s, c, m)$ be the statement that student s has class standing c and is majoring in m. The variable s ranges over students in the class, the variable c ranges over the four class standings, and the variable m ranges over all possible majors.

a) The proposition is $\exists s \exists m P(s, \text{junior}, m)$. It is true from the given information.

b) The proposition is $\forall s \exists c P(s, c, \text{computer science})$. This is false, since there are some mathematics majors.

c) The proposition is $\exists s \exists c \exists m \big(P(s, c, m) \land (c \neq \text{junior}) \land (m \neq \text{mathematics})\big)$. This is true, since there is a sophomore majoring in computer science.

d) The proposition is $\forall s \big(\exists c P(s, c, \text{computer science}) \lor \exists m P(s, \text{sophomore}, m)\big)$. This is false, since there is a freshman mathematics major.

e) The proposition is $\exists m \forall c \exists s P(s, c, m)$. This is false. It cannot be that m is mathematics, since there is no senior mathematics major, and it cannot be that m is computer science, since there is no freshman computer science major. Nor, of course, can m be any other major.

20. **a)** $\forall x \forall y((x < 0) \land (y < 0) \rightarrow (xy > 0))$ **b)** $\forall x \forall y((x > 0) \land (y > 0) \rightarrow ((x + y)/2 > 0))$

c) What does "necessarily" mean in this context? The best explanation is to assert that a certain universal conditional statement is not true. So we have $\neg \forall x \forall y((x < 0) \land (y < 0) \rightarrow (x - y < 0))$. Note that we do not want to put the negation symbol inside (it is not true that the difference of two negative integers is never negative), nor do we want to negate just the conclusion (it is not true that the sum is always nonnegative). We could rewrite our solution by passing the negation inside, obtaining $\exists x \exists y((x < 0) \land (y < 0) \land (x - y \geq 0))$.

d) $\forall x \forall y\,(|x + y| \leq |x| + |y|)$

28. **a)** true (let $y = x^2$) **b)** false (no such y exists if x is negative) **c)** true (let $x = 0$)

d) false (the commutative law for addition always holds) **e)** true (let $y = 1/x$)

f) false (the reciprocal of y depends on y—there is not one x that works for all y) **g)** true (let $y = 1 - x$)

h) false (this system of equations is inconsistent)

i) false (this system has only one solution; if $x = 0$, for example, then no y satisfies $y = 2 \land -y = 1$)

j) true (let $z = (x + y)/2$)

34. The logical expression is asserting that the domain consists of at most two members. (It is saying that whenever you have two unequal objects, any object has to be one of those two. Note that this is vacuously true for domains with one element.) Therefore any domain having one or two members will make it true (such as the female members of the United States Supreme Court in 2005), and any domain with more than two members will make it false (such as all members of the United States Supreme Court in 2005).

1.6

6. Let r be the proposition "It rains," let f be the proposition "It is foggy," let s be the proposition "The sailing race will be held," let l be the proposition "The life saving demonstration will go on," and let t be the proposition "The trophy will be awarded." We are given premises $(\neg r \vee \neg f) \to (s \wedge l)$, $s \to t$, and $\neg t$. We want to conclude r. We set up the proof in two columns, with reasons, as in Example 6. Note that it is valid to replace subexpressions by other expressions logically equivalent to them.

Step	**Reason**
1. $\neg t$	Hypothesis
2. $s \to t$	Hypothesis
3. $\neg s$	Modus tollens using (1) and (2)
4. $(\neg r \vee \neg f) \to (s \wedge l)$	Hypothesis
5. $(\neg(s \wedge l)) \to \neg(\neg r \vee \neg f)$	Contrapositive of (4)
6. $(\neg s \vee \neg l) \to (r \wedge f)$	De Morgan's law and double negative
7. $\neg s \vee \neg l$	Addition, using (3)
8. $r \wedge f$	Modus ponens using (6) and (7)
9. r	Simplification using (8)

10. **a)** If we use modus tollens starting from the back, then we conclude that I am not sore. Another application of modus tollens then tells us that I did not play hockey.

b) We really can't conclude anything specific here.

c) By universal instantiation, we conclude from the first conditional statement by modus ponens that dragonflies have six legs, and we conclude by modus tollens that spiders are not insects. We could say using existential generalization that, for example, there exists a non-six-legged creature that eats a six-legged creature, and that there exists a non-insect that eats an insect.

d) We can apply universal instantiation to the conditional statement and conclude that if Homer (respectively, Maggie) is a student, then he (she) has an Internet account. Now modus tollens tells us that Homer is not a student. There are no conclusions to be drawn about Maggie.

e) The first conditional statement is that if x is healthy to eat, then x does not taste good. Universal instantiation and modus ponens therefore tell us that tofu does not taste good. The third sentence says that if you eat x, then x tastes good. Therefore the fourth hypothesis already follows (by modus tollens) from the first three. No conclusions can be drawn about cheeseburgers from these statements.

f) By disjunctive syllogism, the first two hypotheses allow us to conclude that I am hallucinating. Therefore by modus ponens we know that I see elephants running down the road.

14. In each case we set up the proof in two columns, with reasons, as in Example 6.

a) Let $c(x)$ be "x is in this class," let $r(x)$ be "x owns a red convertible," and let $t(x)$ be "x has gotten a speeding ticket." We are given premises $c(\text{Linda})$, $r(\text{Linda})$, $\forall x(r(x) \to t(x))$, and we want to conclude $\exists x(c(x) \wedge t(x))$.

Step	Reason
1. $\forall x(r(x) \to t(x))$	Hypothesis
2. $r(\text{Linda}) \to t(\text{Linda})$	Universal instantiation using (1)
3. $r(\text{Linda})$	Hypothesis
4. $t(\text{Linda})$	Modus ponens using (2) and (3)
5. $c(\text{Linda})$	Hypothesis
6. $c(\text{Linda}) \wedge t(\text{Linda})$	Conjunction using (4) and (5)
7. $\exists x(c(x) \wedge t(x))$	Existential generalization using (6)

b) Let $r(x)$ be "r is one of the five roommates listed," let $d(x)$ be "x has taken a course in discrete mathematics," and let $a(x)$ be "x can take a course in algorithms." We are given premises $\forall x(r(x) \to d(x))$ and $\forall x(d(x) \to a(x))$, and we want to conclude $\forall x(r(x) \to a(x))$. In what follows y represents an arbitrary person.

Step	Reason
1. $\forall x(r(x) \to d(x))$	Hypothesis
2. $r(y) \to d(y)$	Universal instantiation using (1)
3. $\forall x(d(x) \to a(x))$	Hypothesis
4. $d(y) \to a(y)$	Universal instantiation using (3)
5. $r(y) \to a(y)$	Hypothetical syllogism using (2) and (4)
6. $\forall x(r(x) \to a(x))$	Universal generalization using (5)

c) Let $s(x)$ be "x is a movie produced by Sayles," let $c(x)$ be "x is a movie about coal miners," and let $w(x)$ be "movie x is wonderful." We are given premises $\forall x(s(x) \to w(x))$ and $\exists x(s(x) \wedge c(x))$, and we want to conclude $\exists x(c(x) \wedge w(x))$. In our proof, y represents an unspecified particular movie.

Step	Reason
1. $\exists x(s(x) \wedge c(x))$	Hypothesis
2. $s(y) \wedge c(y)$	Existential instantiation using (1)
3. $s(y)$	Simplification using (2)
4. $\forall x(s(x) \to w(x))$	Hypothesis
5. $s(y) \to w(y)$	Universal instantiation using (4)
6. $w(y)$	Modus ponens using (3) and (5)
7. $c(y)$	Simplification using (2)
8. $w(y) \wedge c(y)$	Conjunction using (6) and (7)
9. $\exists x(c(x) \wedge w(x))$	Existential generalization using (8)

d) Let $c(x)$ be "x is in this class," let $f(x)$ be "x has been to France," and let $l(x)$ be "x has visited the Louvre." We are given premises $\exists x(c(x) \wedge f(x))$, $\forall x(f(x) \to l(x))$, and we want to conclude $\exists x(c(x) \wedge l(x))$.

In our proof, y represents an unspecified particular person.

Step	Reason
1. $\exists x(c(x) \wedge f(x))$	Hypothesis
2. $c(y) \wedge f(y)$	Existential instantiation using (1)
3. $f(y)$	Simplification using (2)
4. $c(y)$	Simplification using (2)
5. $\forall x(f(x) \to l(x))$	Hypothesis
6. $f(y) \to l(y)$	Universal instantiation using (5)
7. $l(y)$	Modus ponens using (3) and (6)
8. $c(y) \wedge l(y)$	Conjunction using (4) and (7)
9. $\exists x(c(x) \wedge l(x))$	Existential generalization using (8)

24. Steps 3 and 5 are incorrect; simplification applies to conjunctions, not disjunctions.

34. Let us use the following letters to stand for the relevant propositions: d for "logic is difficult"; s for "many students like logic"; and e for "mathematics is easy." Then the assumptions are $d \vee \neg s$ and $e \rightarrow \neg d$. Note that the first of these is equivalent to $s \rightarrow d$, since both forms are false if and only if s is true and d is false. In addition, let us note that the second assumption is equivalent to its contrapositive, $d \rightarrow \neg e$. And finally, by combining these two conditional statements, we see that $s \rightarrow \neg e$ also follows from our assumptions.

a) Here we are asked whether we can conclude that $s \rightarrow \neg e$. As we noted above, the answer is yes, this conclusion is valid.

b) The question concerns $\neg e \rightarrow \neg s$. This is equivalent to its contrapositive, $s \rightarrow e$. That doesn't seem to follow from our assumptions, so let's find a case in which the assumptions hold but this conditional statement does not. This conditional statement fails in the case in which s is true and e is false. If we take d to be true as well, then both of our assumptions are true. Therefore this conclusion is not valid.

c) The issue is $\neg e \vee d$, which is equivalent to the conditional statement $e \rightarrow d$. This does *not* follow from our assumptions. If we take d to be false, e to be true, and s to be false, then this proposition is false but our assumptions are true.

d) The issue is $\neg d \vee \neg e$, which is equivalent to the conditional statement $d \rightarrow \neg e$. We noted above that this validly follows from our assumptions.

e) This sentence says $\neg s \rightarrow (\neg e \vee \neg d)$. The only case in which this is false is when s is false and both e and d are true. But in this case, our assumption $e \rightarrow \neg d$ is also violated. Therefore, in all cases in which the assumptions hold, this statement holds as well, so it *is* a valid conclusion.

10. A rational number is a number that can be written in the form x/y where x and y are integers and $y \neq 0$. Suppose that we have two rational numbers, say a/b and c/d. Then their product is, by the usual rules for multiplication of fractions, $(ac)/(bd)$. Note that both the numerator and the denominator are integers, and that $bd \neq 0$ since b and d were both nonzero. Therefore the product is, by definition, a rational number.

12. This is true. Suppose that a/b is a nonzero rational number and that x is an irrational number. We must prove that the product xa/b is also irrational. We give a proof by contradiction. Suppose that xa/b were rational. Since $a/b \neq 0$, we know that $a \neq 0$, so b/a is also a rational number. Let us multiply this rational number b/a by the assumed rational number xa/b. By Exercise 26, the product is rational. But the product is $(b/a)(xa/b) = x$, which is irrational by hypothesis. This is a contradiction, so in fact xa/b must be irrational, as desired.

22. We give a proof by contradiction. Suppose that we don't get a pair of blue socks or a pair of black socks. Then we drew at most one of each color. This accounts for only two socks. But we are drawing three socks. Therefore our supposition that we did not get a pair of blue socks or a pair of black socks is incorrect, and our proof is complete.

Show by direct proof that \$k (where k is positive integer k>1) can be made up using \$3 and \$2 bills.

Proof: Assume k >1 is an integer. If k is even then there is some positive integer *l* such that k = 2*l* and so \$k can be made using *l* \$2 bills. If k is odd then there is some integer *m* such that k = 2*m*+1=2(m-1)+3. In this case \$k can be made using *m*-1 \$2 bills and 1 \$3 bill. Thus in every case \$k can be made using \$2 and \$3 bills.

18. Given r, let a be the closest integer to r less than r, and let b be the closest integer to r greater than r. In the notation to be introduced in Section 2.3, $a = \lfloor r \rfloor$ and $b = \lceil r \rceil$. In fact, $b = a + 1$. Clearly the distance between r and any integer other than a or b is greater than 1 so cannot be less than 1/2. Furthermore, since r is irrational, it cannot be exactly half-way between a and b, so exactly one of $r - a < 1/2$ and $b - r < 1/2$ holds.

30. If $|y| \geq 2$, then $2x^2 + 5y^2 \geq 2x^2 + 20 \geq 20$, so the only possible values of y to try are 0 and ± 1. In the former case we would be looking for solutions to $2x^2 = 14$ and in the latter case to $2x^2 = 9$. Clearly there are no integer solutions to these equations, so there are no solutions to the original equation.

nine 1's at some point. But in the step before that each adjacent pair of bits must have been different; in other words, they must have alternated 0, 1, 0, 1, and so on. This is impossible with an odd number of bits. This contradiction shows that we can never get nine 0's.

Find two positive integers M and N such that $M^2 – N^2 = 5213 \times 4029$. (Hint: Use the fact that $M^2 – N^2 = (M + N) * (M – N)$.)

Solution: Using the hint we set M+N=5213 and M-N=4029. Solving this system gives us the solution M=4621 and N=592.

Give an example of a ten digit number X that can't be written as $X = M^2 – N^2$. (Hint: The number 1000000007 is prime. Argue that 2 x 1000000007 can't be a difference of two perfect squares.)

Solution: Prof by contradiction. Assume that 2 x 1000000007 can be written as the difference of perfect squares. Using the hint as well as the hint from the previous problem we assume that 2 x 1000000007 $=M^2 - N^2$ =(M-N)(M+N). Then M-N and M+N must be even and so there are integers k and l such that M+N=2k and M-N=2l with k not equal to l. Furthermore 2 x 1000000007/(M+N) is an integer. However this wold imply
2 x 1000000007/(M+N) = 2 x 1000000007/2k =1000000007/k is an integer and 1000000007 is dividable by k . A similar argument shows that 1000000007 would also be diviable by l since k and l are different they cannot both be equal to one which is a contradiction since 1000000007 is odd. Thus 2 x 1000000007 must be a 10 digit number that cannot be written as the difference of squares.

10. a) true **b)** true **c)** false—see part **(a)** **d)** true
e) true—the one element in the set on the left is an element of the set on the right, and the sets are not equal
f) true—similar to part **(e)** **g)** false—the two sets are equal

2.1

24. a) The power set of every set includes at least the empty set, so the power set cannot be empty. Thus $\emptyset$ is not the power set of any set.
b) This is the power set of $\{a\}$.
c) This set has three elements. Since 3 is not a power of 2, this set cannot be the power set of any set.
d) This is the power set of $\{a, b\}$.

40. The only difference between $(A \times B) \times (C \times D)$ and $A \times (B \times C) \times D$ is parentheses, so for all practical purposes one can think of them as essentially the same thing. By Definition 8, the elements of $(A \times B) \times (C \times D)$ consist of ordered pairs (x, y), where $x \in A \times B$ and $y \in C \times D$, so the typical element of $(A \times B) \times (C \times D)$ looks like $((a,b),(c,d))$. By Definition 9, the elements of $A \times (B \times C) \times D$ consist of 3-tuples (a, x, d), where $a \in A$, $d \in D$, and $x \in B \times C$, so the typical element of $A \times (B \times C) \times D$ looks like $(a,(b,c),d)$. The structures $((a,b),(c,d))$ and $(a,(b,c),d)$ are different, even if they convey exactly the same information (the first is a pair, and the second is a 3-tuple). To be more precise, there is a natural one-to-one correspondence between $(A \times B) \times (C \times D)$ and $A \times (B \times C) \times D$ given by $((a,b),(c,d)) \leftrightarrow (a,(b,c),d)$.

42. a) There is a real number whose cube is -1. This is true, since $x = -1$ is a solution.

b) There is an integer such that the number obtained by adding 1 to it is greater than the integer. This is true—in fact, every integer satisfies this statement.

c) For every integer, the number obtained by subtracting 1 is again an integer. This is true.

d) The square of every integer is an integer. This is true.

2.2

18. a) Suppose that $x \in A \cup B$. Then either $x \in A$ or $x \in B$. In either case, certainly $x \in A \cup B \cup C$. This establishes the desired inclusion.

b) Suppose that $x \in A \cap B \cap C$. Then x is in all three of these sets. In particular, it is in both A and B and therefore in $A \cap B$, as desired.

c) Suppose that $x \in (A - B) - C$. Then x is in $A - B$ but not in C. Since $x \in A - B$, we know that $x \in A$ (we also know that $x \notin B$, but that won't be used here). Since we have established that $x \in A$ but $x \notin C$, we have proved that $x \in A - C$.

d) To show that the set given on the left-hand side is empty, it suffices to assume that x is some element in that set and derive a contradiction, thereby showing that no such x exists. So suppose that $x \in (A-C) \cap (C-B)$. Then $x \in A - C$ and $x \in C - B$. The first of these statements implies by definition that $x \notin C$, while the second implies that $x \in C$. This is impossible, so our proof by contradiction is complete.

e) To establish the equality, we need to prove inclusion in both directions. To prove that $(B-A) \cup (C-A) \subseteq (B \cup C) - A$, suppose that $x \in (B-A) \cup (C-A)$. Then either $x \in (B-A)$ or $x \in (C-A)$. Without loss of

generality, assume the former (the proof in the latter case is exactly parallel.) Then $x \in B$ and $x \notin A$. From the first of these assertions, it follows that $x \in B \cup C$. Thus we can conclude that $x \in (B \cup C) - A$, as desired. For the converse, that is, to show that $(B \cup C) - A \subseteq (B - A) \cup (C - A)$, suppose that $x \in (B \cup C) - A$. This means that $x \in (B \cup C)$ and $x \notin A$. The first of these assertions tells us that either $x \in B$ or $x \in C$. Thus either $x \in B - A$ or $x \in C - A$. In either case, $x \in (B - A) \cup (C - A)$. (An alternative proof could be given by using Venn diagrams, showing that both sides represent the same region.)

30. **a)** We cannot conclude that $A = B$. For instance, if A and B are both subsets of C, then this equation will always hold, and A need not equal B.

b) We cannot conclude that $A = B$; let $C = \emptyset$, for example.

c) By putting the two conditions together, we *can* now conclude that $A = B$. By symmetry, it suffices to prove that $A \subseteq B$. Suppose that $x \in A$. There are two cases. If $x \in C$, then $x \in A \cap C = B \cap C$, which forces $x \in B$. On the other hand, if $x \notin C$, then because $x \in A \cup C = B \cup C$, we must have $x \in B$.

50. **a)** As i increases, the sets get smaller: $\cdots \subset A_3 \subset A_2 \subset A_1$. All the sets are subsets of A_1, which is the set of positive integers, $\mathbf{Z}^+$. It follows that $\bigcup_{i=1}^{\infty} A_i = \mathbf{Z}^+$. Every positive integer is excluded from at least one of the sets (in fact from infinitely many), so $\bigcap_{i=1}^{\infty} A_i = \emptyset$.

b) All the sets are subsets of the set of natural numbers $\mathbf{N}$ (the nonnegative integers). The number 0 is in each of the sets, and every positive integer is in exactly one of the sets, so $\bigcup_{i=1}^{\infty} A_i = \mathbf{N}$ and $\bigcap_{i=1}^{\infty} A_i = \{0\}$.

c) As i increases, the sets get larger: $A_1 \subset A_2 \subset A_3 \cdots$. All the sets are subsets of the set of positive real numbers $\mathbf{R}^+$, and every positive real number is included eventually, so $\bigcup_{i=1}^{\infty} A_i = \mathbf{R}^+$. Because A_1 is a subset of each of the others, $\bigcap_{i=1}^{\infty} A_i = A_1 = (0, 1)$ (the interval of all real numbers between 0 and 1, exclusive).

d) This time, as in part **(a)**, the sets are getting smaller as i increases: $\cdots \subset A_3 \subset A_2 \subset A_1$. Because A_1 includes all the others, $\bigcup_{i=1}^{\infty} A_1 = (1, \infty)$ (all real numbers greater than 1). Every number eventually gets excluded as i increases, so $\bigcap_{i=1}^{\infty} A_i = \emptyset$. Notice that ∞ is not a real number, so we cannot write $\bigcap_{i=1}^{\infty} A_i = \{\infty\}$.

52. **a)** 00 1110 0000 **b)** 10 1001 0001 **c)** 01 1100 1110

2.3

10. **a)** This is one-to-one. **b)** This is not one-to-one, since b is the image of both a and b.
c) This is not one-to-one, since d is the image of both a and d.

14. **a)** This is clearly onto, since $f(0, -n) = n$ for every integer n.

b) This is not onto, since, for example, 2 is not in the range. To see this, if $m^2 - n^2 = (m - n)(m + n) = 2$, then m and n must have same parity (both even or both odd). In either case, both $m - n$ and $m + n$ are then even, so this expression is divisible by 4 and hence cannot equal 2.

c) This is clearly onto, since $f(0, n - 1) = n$ for every integer n.

d) This is onto. To achieve negative values we set $m = 0$, and to achieve nonnegative values we set $n = 0$.

e) This is not onto, for the same reason as in part **(b)**. In fact, the range here is clearly a subset of the range in that part.

38. Forming the compositions we have $(f \circ g)(x) = acx + ad + b$ and $(g \circ f)(x) = cax + cb + d$. These are equal if and only if $ad + b = cb + d$. In other words, equality holds for all 4-tuples (a, b, c, d) for which $ad + b = cb + d$.

42. **a)** The answer is the set of all solutions to $x^2 = 1$, namely $\{1, -1\}$.

b) In order for x^2 to be strictly between 0 and 1, we need x to be either strictly between 0 and 1 or strictly between -1 and 0. Therefore the answer is $\{x \mid -1 < x < 0 \vee 0 < x < 1\}$.

c) In order for x^2 to be greater than 4, we need either $x > 2$ or $x < -2$. Therefore the answer is $\{x \mid x > 2 \vee x < -2\}$.