Binomial Coefficients and Identities

Section Summary

- The Binomial Theorem
- Pascal's Identity and Triangle
- Other Identities Involving Binomial Coefficients

Powers of Binomial Expressions

Definition: A *binomial* expression is the sum of two terms, such as $x + y$. (More generally, these terms can be products of constants and variables.)

- We can use counting principles to find the coefficients in the expansion of $(x + y)^n$ where n is a positive integer.
- To illustrate this idea, we first look at the process of expanding $(x + y)^3$.
- $(x + y)(x + y)(x + y)$ expands into a sum of terms that are the product of a term from each of the three sums.
- Terms of the form x^3, x^2y, $x y^2$, y^3 arise. The question is what are the coefficients?
  - To obtain x^3, an x must be chosen from each of the sums. There is only one way to do this. So, the coefficient of x^3 is 1.
  - To obtain x^2y, an x must be chosen from two of the sums and a y from the other. There are $\binom{3}{2}$ ways to do this and so the coefficient of x^2y is 3.
  - To obtain xy^2, an x must be chosen from of the sums and a y from the other two . There are $\binom{3}{1}$ ways to do this and so the coefficient of xy^2 is 3.
  - To obtain y^3, a y must be chosen from each of the sums. There is only one way to do this. So, the coefficient of y^3 is 1.
- We have used a counting argument to show that $(x + y)^3 = x^3 + 3x^2y + 3x y^2 + y^3$.
- Next we present the binomial theorem gives the coefficients of the terms in the expansion of $(x + y)^n$.

Binomial Theorem

Binomial Theorem: Let x and y be variables, and n a nonnegative integer. Then:

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j}y^j = \binom{n}{0} x^n + \binom{n}{1} x^{n-1}y + \cdots + \binom{n}{n-1} xy^{n-1} + \binom{n}{n} y^n.$$

Proof: We use combinatorial reasoning . The terms in the expansion of $(x + y)^n$ are of the form $x^{n-j}y^j$ for $j = 0,1,2,\ldots,n$. To form the term $x^{n-j}y^j$, it is necessary to choose $n-j$ xs from the n sums. Therefore, the coefficient of $x^{n-j}y^j$ is $\binom{n}{n-j}$ which equals $\binom{n}{j}$. ◀

Using the Binomial Theorem

Example: What is the coefficient of $x^{12}y^{13}$ in the expansion of $(2x - 3y)^{25}$?

Solution: We view the expression as $(2x + (-3y))^{25}$. By the binomial theorem

$$(2x + (-3y))^{25} = \sum_{j=0}^{25} \binom{25}{j} (2x)^{25-j}(-3y)^j.$$

Consequently, the coefficient of $x^{12}y^{13}$ in the expansion is obtained when $j = 13$.

$$\binom{25}{13} 2^{12}(-3)^{13} = -\frac{25!}{13!12!}2^{12}3^{13}.$$

A Useful Identity

Corollary 1: With $n \geq 0$, $$\sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

Proof (*using binomial theorem*): With $x = 1$ and $y = 1$, from the binomial theorem we see that:

$$2^n = (1+1)^n = \sum_{k=0}^{n} \binom{n}{k} 1^k 1^{(n-k)} = \sum_{k=0}^{n} \binom{n}{k}. \quad \blacktriangleleft$$

Proof (*combinatorial*): Consider the subsets of a set with n elements. There are $\binom{n}{0}$ subsets with zero elements, $\binom{n}{1}$ with one element, $\binom{n}{2}$ with two elements, ..., and $\binom{n}{n}$ with n elements. Therefore the total is $$\sum_{k=0}^{n} \binom{n}{k}.$$

Since, we know that a set with n elements has 2^n subsets, we conclude:
$$\sum_{k=0}^{n} \binom{n}{k} = 2^n. \quad \blacktriangleleft$$

Pascal's Identity

Pascal's Identity: If n and k are integers with $n \geq k \geq 0$, then

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}.$$

Proof (*combinatorial*): Let T be a set where $|T| = n + 1$, $a \in T$, and $S = T - \{a\}$. There are $\binom{n+1}{k}$ subsets of T containing k elements. Each of these subsets either:

- contains a with $k - 1$ other elements, or
- contains k elements of S and not a.

There are

- $\binom{n}{k-1}$ subsets of k elements that contain a, since there are $\binom{n}{k-1}$ subsets of $k - 1$ elements of S,
- $\binom{n}{k}$ subsets of k elements of T that do not contain a, because there are $\binom{n}{k}$ subsets of k elements of S.

Hence,

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}. \blacktriangleleft$$

See Exercise 19 *for an algebraic proof.*

Pascal's Triangle

The *n*th row in the triangle consists of the binomial coefficients $\binom{n}{k}$, $k = 0,1,\ldots,n$.

$$
\begin{array}{c}
\binom{0}{0} \\
\binom{1}{0} \quad \binom{1}{1} \\
\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2} \\
\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3} \\
\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4} \\
\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5} \\
\binom{6}{0} \quad \binom{6}{1} \quad \binom{6}{2} \quad \binom{6}{3} \quad \binom{6}{4} \quad \binom{6}{5} \quad \binom{6}{6} \\
\binom{7}{0} \quad \binom{7}{1} \quad \binom{7}{2} \quad \binom{7}{3} \quad \binom{7}{4} \quad \binom{7}{5} \quad \binom{7}{6} \quad \binom{7}{7} \\
\binom{8}{0} \quad \binom{8}{1} \quad \binom{8}{2} \quad \binom{8}{3} \quad \binom{8}{4} \quad \binom{8}{5} \quad \binom{8}{6} \quad \binom{8}{7} \quad \binom{8}{8} \\
\cdots
\end{array}
$$

(a)

By Pascal's identity:

$$\binom{6}{4} + \binom{6}{5} = \binom{7}{5}$$

$$
\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad 2 \quad 1 \\
1 \quad 3 \quad 3 \quad 1 \\
1 \quad 4 \quad 6 \quad 4 \quad 1 \\
1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \\
1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1 \\
1 \quad 7 \quad 21 \quad 35 \quad 35 \quad 21 \quad 7 \quad 1 \\
1 \quad 8 \quad 28 \quad 56 \quad 70 \quad 56 \quad 28 \quad 8 \quad 1 \\
\cdots
\end{array}
$$

(b)

By Pascal's identity, adding two adjacent bionomial coefficients results is the binomial coefficient in the next row between these two coefficients.

11. How many bit strings of length 10 contain

a) exactly four 1s?

b) at most four 1s?

c) at least four 1s?

d) an equal number of 0s and 1s?

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d) an equal number of 0s and 1s?

Solution: (a) C(10, 4) or $\binom{10}{4}$ = 10!/ (6! x 4!) = $\frac{7.8.9.10}{24}$

(b) C(10,0) + C(10, 1) + C(10, 2) + C(10, 3)

(c) $\sum_{j=4}^{10} C(10, j)$

(d) C(10, 5)

22. How many permutations of the letters $ABCDEFGH$ contain

a) the string ED?

b) the string CDE?

c) the strings BA and FGH?

d) the strings AB, DE, and GH?

e) the strings CAB and BED?

f) the strings BCA and ABF?

22. How many permutations of the letters *ABCDEFGH* contain

a) the string *ED*?
b) the string *CDE*?
c) the strings *BA* and *FGH*?
d) the strings *AB*, *DE*, and *GH*?
e) the strings *CAB* and *BED*?
f) the strings *BCA* and *ABF*?

Solution: (a) suppose the string ED occurs in position 1, then the remaining letters can be filled in 6! ways. The same is true if ED occurs in any of the positions 1, 2, …, 7. All these sets of permutations are disjoint and so the total number is 7 x 6! = 7!

A much simpler way to do this is to create a new "super letter" ED so there are now 7 letters, A, B, C, ED, F, G and H.

(d) AB, DE and GH: There are 6 different orders in which these can occur. The remaining 2 letters can be placed in four bins independently in C(4+2-1, 2) = 10 ways, and thus in 20 ways if order is considered.

So the total number is 6 x 20 = 120.

Again, a much simpler way to solve this is to glue the letters to create 5 new letters AB, DE, GH, C and F. There are 5! ways they can be placed.

30. Seven women and nine men are on the faculty in the mathematics department at a school.

a) How many ways are there to select a committee of five members of the department if at least one woman must be on the committee?

b) How many ways are there to select a committee of five members of the department if at least one woman and at least one man must be on the committee?

30. Seven women and nine men are on the faculty in the mathematics department at a school.

a) How many ways are there to select a committee of five members of the department if at least one woman must be on the committee?

b) How many ways are there to select a committee of five members of the department if at least one woman and at least one man must be on the committee?

Solution:

a) There are $C(16,5)$ ways to select a committee if there are no restrictions. There are $C(9,5)$ ways to select a committee from just the 9 men. Therefore there are $C(16,5) - C(9,5) = 4368 - 126 = 4242$ committees with at least one woman.

b) Answer:

$C(16, 5) - C(9, 5) - C(7, 5) = 4368 - 126 - 21 = 4221$

36) How many bit strings contain exactly five 0s and 14 1s if every 0 must be immediately followed by two 1s?

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Answer: We now have patterns A = 011 (repeated 5 times) and the remaining 4 ones. Thus, we are creating binary strings over {1, A} of length 9 with four 1's and 5 A's.

The number of such strings is C(9, 4) = 126.

42. Find a formula for the number of ways to seat r of n people around a circular table, where seatings are considered the same if every person has the same two neighbors without regard to which side these neighbors are sitting on.

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Solution: The requirement states that two seating are the same if one can be obtained from the other by a cyclic rotation or one of them can be obtained by reflecting on a mirror.

There are totally C(n, r) ways to select r people, by cycling shifting generates r equivalent groups. Thus, there are C(n, r)/r circular arrangements (without discounting for reflection).

Reflection will reduce this number by a factor of 2 since each arrangement can be reflected to give an equivalent arrangement.

Thus the answer is C(n, r)/ (2 * r)

9. What is the coefficient of $x^{101}y^{99}$ in the expansion of $(2x-3y)^{200}$?

Answer: C(200, 101) * 2^{101} *$(-3)^{99}$

14. Show that if n is a positive integer, then $1 = \binom{n}{0} < \binom{n}{1} < \cdots < \binom{n}{\lfloor n/2 \rfloor} = \binom{n}{\lceil n/2 \rceil} > \cdots > \binom{n}{n-1} > \binom{n}{n} = 1$.

PROOF:

Using the factorial formulae for computing binomial coefficients, we see that $\binom{n}{k-1} = \frac{k}{n-k+1}\binom{n}{k}$. If $k \le n/2$, then $\frac{k}{n-k+1} < 1$, so the "less than" signs are correct. Similarly, if $k > n/2$, then $\frac{k}{n-k+1} > 1$, so the "greater than" signs are correct. The middle equality is Corollary 2 in Section 6.3, since $\lfloor n/2 \rfloor + \lceil n/2 \rceil = n$. The equalities at the ends are clear.

22. Prove the identity $\binom{n}{r}\binom{r}{k} = \binom{n}{k}\binom{n-k}{r-k}$, whenever n, r, and k are nonnegative integers with $r \le n$ and $k \le r$,

a) using a combinatorial argument.

b) using an argument based on the formula for the number of r-combinations of a set with n elements.

22. Prove the identity $\binom{n}{r}\binom{r}{k} = \binom{n}{k}\binom{n-k}{r-k}$, whenever n, r, and k are nonnegative integers with $r \leq n$ and $k \leq r$,

a) using a combinatorial argument.

b) using an argument based on the formula for the number of r-combinations of a set with n elements.

a) What is combinatorial argument? To show that two expressions a and b are equal, you should create two sets A and B such that |A| = a and |B| = b, then show that there is a bijective mapping between A and B. Let there be n players. We want to choose a team with r players such that k will actually play and the remaining r-k are reverses. We can use two different process to select: first select the whole team, then pick the reserves from the team. Or, we can select the players, then pick the reserves from the remaining ones.

b) On the one hand,

$$\binom{n}{r}\binom{r}{k} = \frac{n!}{r!(n-r)!} \cdot \frac{r!}{k!(r-k)!} = \frac{n!}{k!(n-r)!(r-k)!},$$

and on the other hand

$$\binom{n}{k}\binom{n-k}{r-k} = \frac{n!}{k!(n-k)!} \cdot \frac{(n-k)!}{(r-k)!(n-r)!} = \frac{n!}{k!(n-r)!(r-k)!}.$$

Generalized Permutations and Combinations

Section Summary

- Permutations with Repetition
- Combinations with Repetition
- Permutations with Indistinguishable Objects
- Distributing Objects into Boxes

Permutations with Repetition

Theorem 1: The number of r-permutations of a set of n objects with repetition allowed is n^r.

Proof: There are n ways to select an element of the set for each of the r positions in the r-permutation when repetition is allowed. Hence, by the product rule there are n^r r-permutations with repetition.

◀

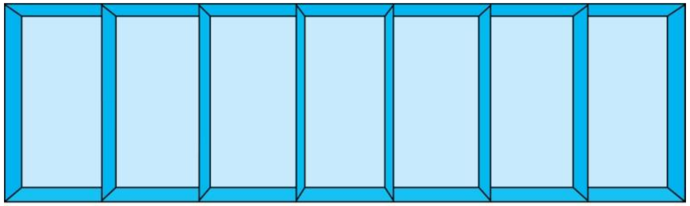
Example: How many strings of length r can be formed from the uppercase letters of the English alphabet?

Solution: The number of such strings is 26^r, which is the number of r-permutations of a set with 26 elements.

Combinations with Repetition

Example: How many ways are there to select five bills from a box containing at least five of each of the following denominations: \$1, \$2, \$5, \$10, \$20, \$50, and \$100?

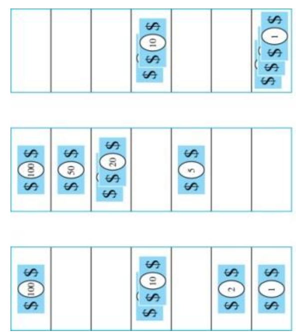
Solution: Place the selected bills in the appropriate position of a cash box illustrated below:



continued →

Combinations with Repetition

- Some possible ways of placing the five bills:



$|\,|\,|\,*\,*\,|\,|\,|\,*\,*\,*$

$*\,|\,*\,|\,*\,*\,|\,|\,*\,|\,|$

$*\,|\,|\,|\,*\,*\,|\,|\,*\,|\,*$

- The number of ways to select five bills corresponds to the number of ways to arrange six bars and five stars in a row.
- This is the number of unordered selections of 5 objects from a set of 11. Hence, there are

$$C(11,5) = \frac{11!}{5!6!} = 462$$

ways to choose five bills with seven types of bills.

Combinations with Repetition

Theorem 2: The number of r-combinations from a set with n elements when repetition of elements is allowed is

$$C(n + r - 1, r) = C(n + r - 1, n - 1).$$

Proof: Each r-combination of a set with n elements with repetition allowed can be represented by a list of $n - 1$ bars and r stars. The bars mark the n cells containing a star for each time the ith element of the set occurs in the combination.

The number of such lists is $C(n + r - 1, r)$, because each list is a choice of the r positions to place the stars, from the total of $n + r - 1$ positions to place the stars and the bars. This is also equal to $C(n + r - 1, n - 1)$, which is the number of ways to place the $n - 1$ bars.

Combinations with Repetition

Example: How many solutions does the equation

$$x_1 + x_2 + x_3 = 11$$

have, where x_1 , x_2 and x_3 are nonnegative integers?

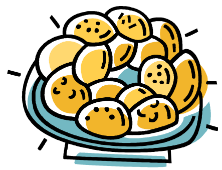
Solution: Each solution corresponds to a way to select 11 items from a set with three elements; x_1 elements of type one, x_2 of type two, and x_3 of type three.

By Theorem 2 it follows that there are

$$C(3+11-1,11) = C(13,11) = C(13,2) = \frac{13 \cdot 12}{1 \cdot 2} = 78$$

solutions.

Combinations with Repetition



Example: Suppose that a cookie shop has four different kinds of cookies. How many different ways can six cookies be chosen?

Solution: The number of ways to choose six cookies is the number of 6-combinations of a set with four elements. By Theorem 2

$$C(9,6) = C(9,3) = \frac{9\cdot 8\cdot 7}{1\cdot 2\cdot 3} = 84$$

is the number of ways to choose six cookies from the four kinds.

Summarizing the Formulas for Counting Permutations and Combinations with and without Repetition

TABLE 1 Combinations and Permutations With and Without Repetition.

Type	*Repetition Allowed?*	*Formula*
r-permutations	No	$\frac{n!}{(n-r)!}$
r-combinations	No	$\frac{n!}{r!\,(n-r)!}$
r-permutations	Yes	n^r
r-combinations	Yes	$\frac{(n+r-1)!}{r!\,(n-1)!}$

How many different combinations of pennies, nickels, dimes, quarters, and half dollars can a piggy bank contain if it has 20 coins in it?

How many different combinations of pennies, nickels, dimes, quarters, and half dollars can a piggy bank contain if it has 20 coins in it?

Solution: There are 5 things to choose from, repetitions allowed, and we want to choose 20 things, order not important.

Therefore by Theorem 2 the answer is $C(5 + 20 - 1, 20) = C(24, 20) = C(24, 4) = 10{,}626$.

18. How many strings of 20-decimal digits are there that contain two 0s, four 1s, three 2s, one 3, two 4s, three 5s, two 7s, and three 9s?

18. How many strings of 20-decimal digits are there that contain two 0s, four 1s, three 2s, one 3, two 4s, three 5s, two 7s, and three 9s?

Solution:

It follows directly from Theorem 3 that the answer is

$$\frac{20!}{2!4!3!1!2!3!2!3!} \approx 5.9 \times 10^{13}\,.$$

Permutations with Indistinguishable Objects

Example: How many different strings can be made by reordering the letters of the word *SUCCESS*.

Solution: There are seven possible positions for the three Ss, two Cs, one U, and one E.

- The three Ss can be placed in $C(7,3)$ different ways, leaving four positions free.
- The two Cs can be placed in $C(4,2)$ different ways, leaving two positions free.
- The U can be placed in $C(2,1)$ different ways, leaving one position free.
- The E can be placed in $C(1,1)$ way.

By the product rule, the number of different strings is:

The reasoning can be generalized to the following theorem. →

$$C(7,3)C(4,2)C(2,1)C(1,1) = \frac{7!}{3!4!} \cdot \frac{4!}{2!2!} \cdot \frac{2!}{1!1!} \cdot \frac{1!}{1!0!} = \frac{7!}{3!2!1!1!} = 420.$$

Permutations with Indistinguishable Objects

Theorem 3: The number of different permutations of n objects, where there are n_1 indistinguishable objects of type 1, n_2 indistinguishable objects of type 2, …., and n_k indistinguishable objects of type k, is:

$$\frac{n!}{n_1!n_2!\cdots n_k!}\ .$$

Proof: By the product rule the total number of permutations is:

$C(n, n_1)\, C(n - n_1, n_2) \cdots C(n - n_1 - n_2 - \cdots - n_k, n_k)$ since:

- The n_1 objects of type one can be placed in the n positions in $C(n, n_1)$ ways, leaving $n - n_1$ positions.
- Then the n_2 objects of type two can be placed in the $n - n_1$ positions in $C(n - n_1, n_2)$ ways, leaving $n - n_1 - n_2$ positions.
- Continue in this fashion, until n_k objects of type k are placed in $C(n - n_1 - n_2 - \cdots - n_k, n_k)$ ways.

The product can be manipulated into the desired result as follows:

$$\frac{n!}{n_1!(n-n_1)!}\frac{(n-n_1)!}{n_2!(n-n_1-n_2!)}\cdots\frac{(n-n_1-\cdots-n_{k-1})!}{n_k!0!} = \frac{n!}{n_1!n_2!\cdots n_k!}\ .$$

Distributing Objects into Boxes

- Many counting problems can be solved by counting the ways objects can be placed in boxes.
  - The objects may be either different from each other (*distinguishable*) or identical (*indistinguishable*).
  - The boxes may be labeled (*distinguishable*) or unlabeled (*indistinguishable*).

Distributing Objects into Boxes

- *Distinguishable objects* and *distinguishable boxes*.
  - There are $n!/(n_1!n_2!\cdots n_k!)$ ways to distribute n distinguishable objects into k distinguishable boxes.
  - (*See Exercises* 47 *and* 48 *for two different proofs*.)
  - Example: There are 52!/(5!5!5!5!32!) ways to distribute hands of 5 cards each to four players.
- *Indistinguishable objects* and *distinguishable boxes*.
  - There are $C(n + r - 1, n - 1)$ ways to place r indistinguishable objects into n distinguishable boxes.
  - Proof based on one-to-one correspondence between n-combinations from a set with k-elements when repetition is allowed and the ways to place n indistinguishable objects into k distinguishable boxes.
  - Example: There are $C(8 + 10 - 1, 10) = C(17,10) = 19,448$ ways to place 10 indistinguishable objects into 8 distinguishable boxes.

Distributing Objects into Boxes

- *Distinguishable objects* and *indistinguishable boxes*.
  - Example: There are 14 ways to put four employees into three indistinguishable offices (*see Example* 10).
  - There is no simple closed formula for the number of ways to distribute n distinguishable objects into j indistinguishable boxes.
  - See the text for a formula involving *Stirling numbers of the second kind*.
- *Indistinguishable objects* and *indistinguishable boxes*.
  - Example: There are 9 ways to pack six copies of the same book into four identical boxes (*see Example* 11).
  - The number of ways of distributing n indistinguishable objects into k indistinguishable boxes equals $p_k(n)$, the number of ways to write n as the sum of at most k positive integers in increasing order.
  - No simple closed formula exists for this number.