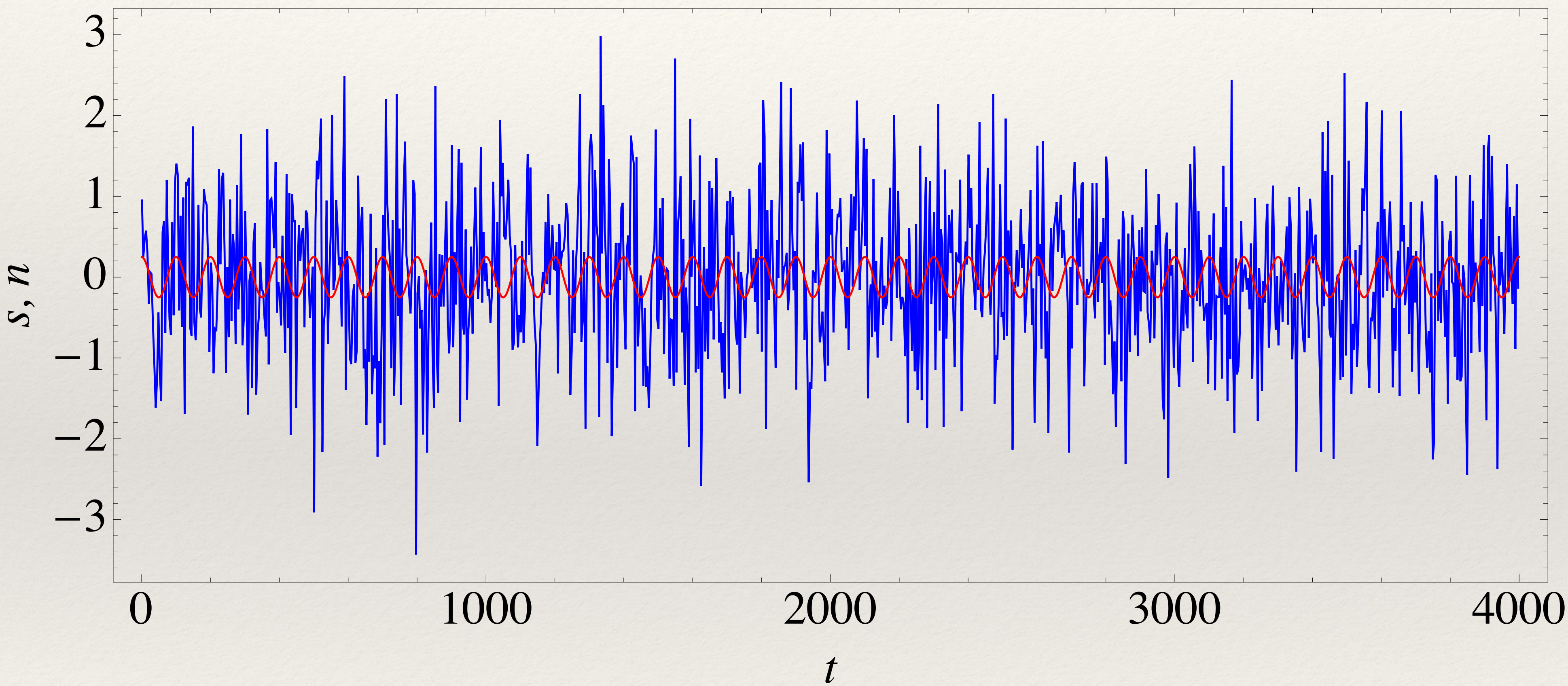
Physics 236 B *(Winter 2013-14):* ***General Relativity****, Week 2, Lecture 2*

Extracting Signal from Noise

Extracting Signal from Noise

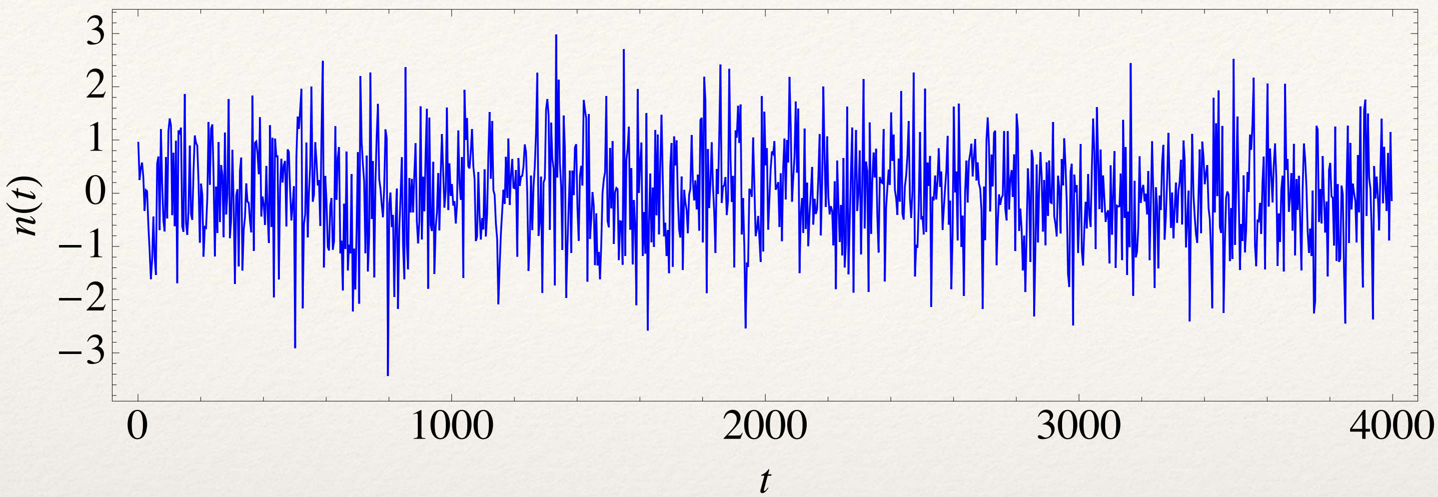
Signals with known shapes can be better extracted from noisy data.

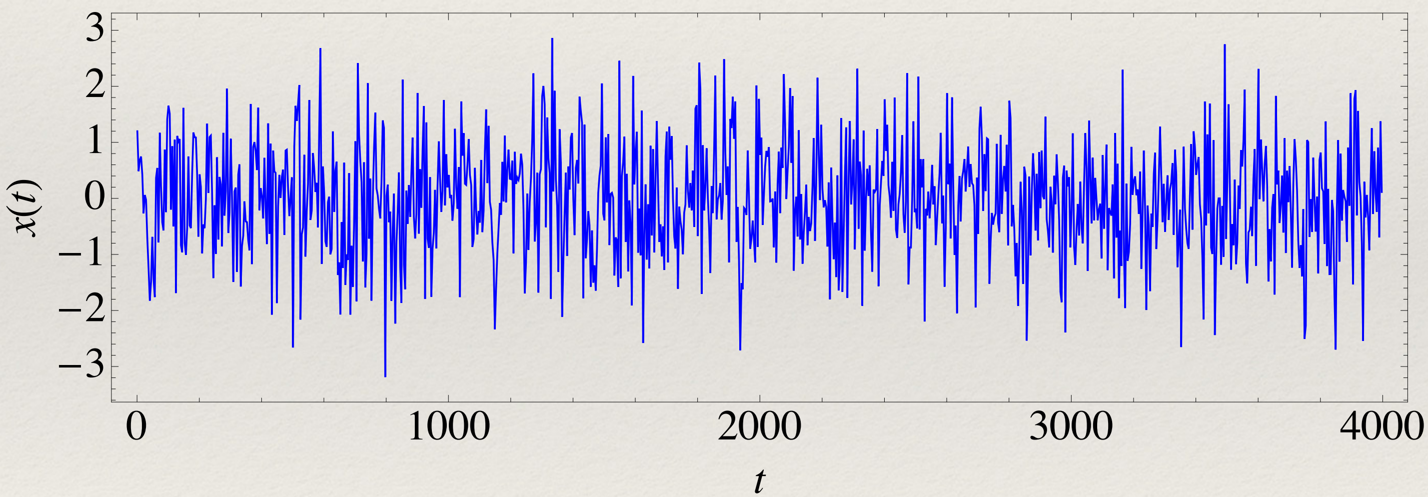
$$x(t) = s(t) + n(t)$$



signal "submerged" in noise: signal amplitude less than noise level

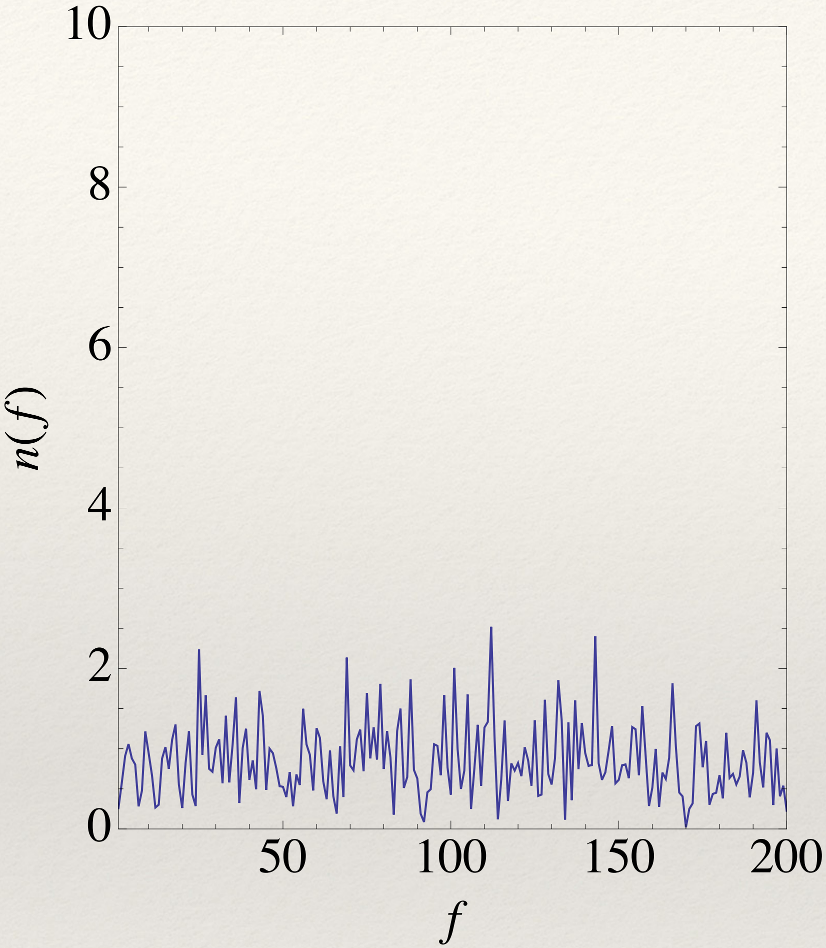
Extracting Signal from Noise

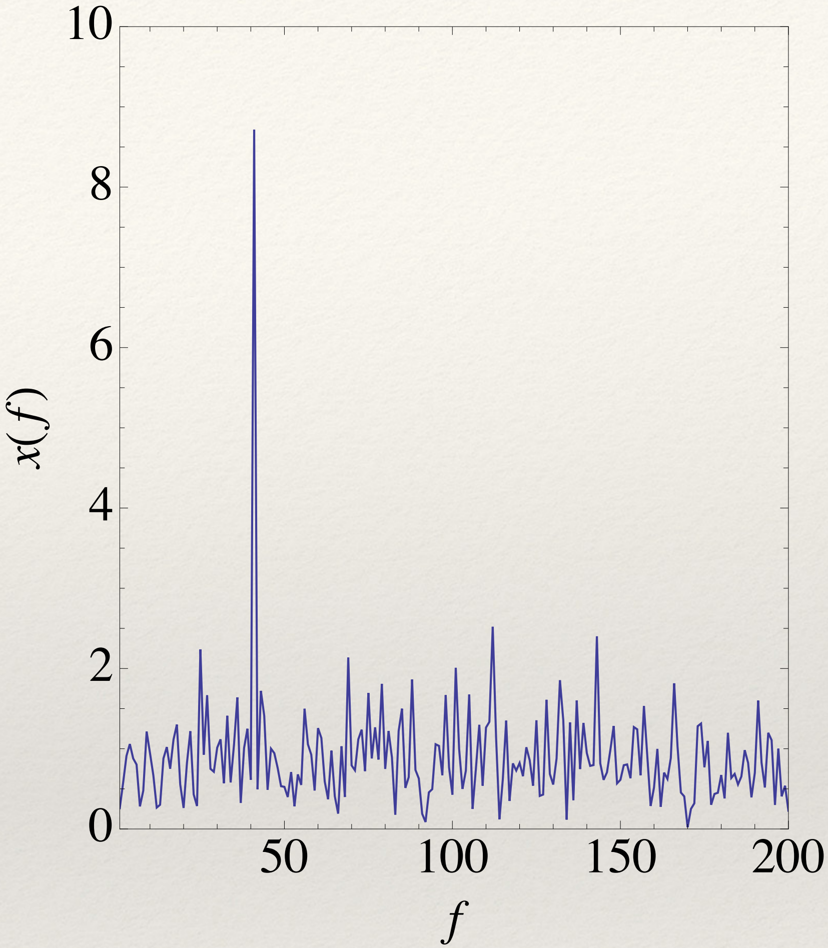




the data looks no different from pure noise

Extracting Signal from Noise

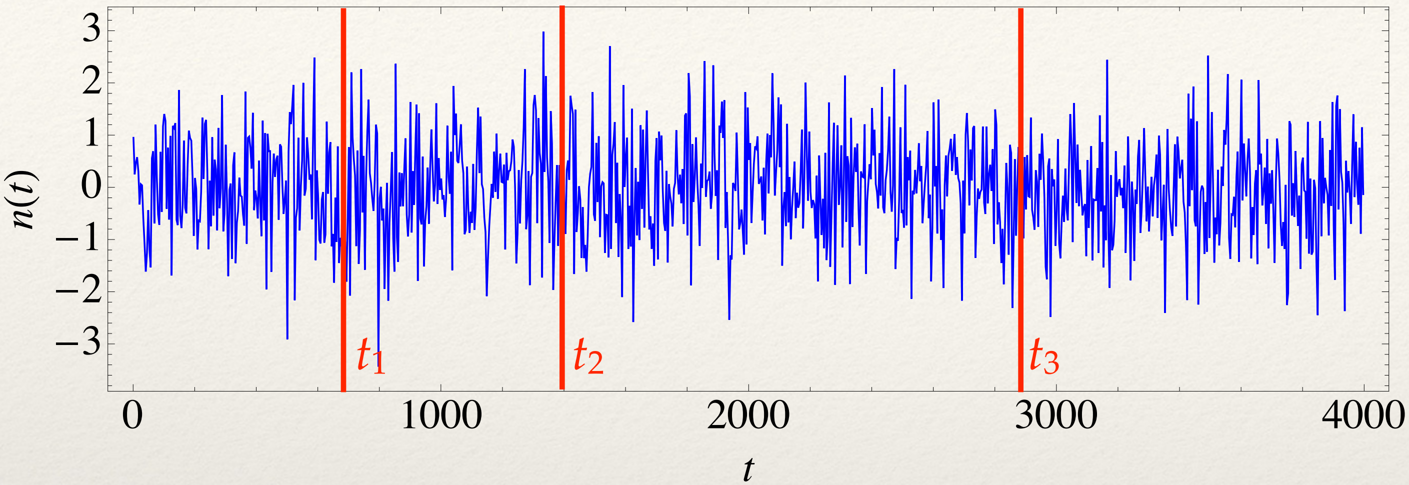




In the frequency domain, signal shows up clearly!

Random Processes

Noise is described by a random process $n(t)$: a series of random variables labeled by t



one "realization" of noise n(t)

A (continuous) random process is fully described by joint probability distributions

$$p_1(t_1, y_1),\ p_2(t_1, y_1; t_2, y_2), \ldots, p_N(t_1, y_1; t_2, y_2; \ldots; t_N y_N)$$

It can be characterized by correlation functions

$$\langle n(t_1) \rangle = \int y_1 p_1(t_1, y_1) dy_1$$

$$\langle n(t_1) n(t_2) \rangle = \int y_1 y_2 p_2(t_1, y_1; t_2, y_2) dy_1 dy_2$$

$$\ldots$$

Random Processes

Stationary random process: distribution the same when shifted in time.

$$p(t_1, y_1; t_2, y_2; \ldots) = p(t_1 - \tau, y_1; t_2 - \tau, y_2; \ldots)$$

In particular, $\langle n(t_1) \rangle = \langle n(t_2) \rangle \quad \langle n(t_1) n(t_2) \rangle = \langle n(0) n(t_2 - t_1) \rangle$

Gaussian Random Process

1. Joint distributions are Gaussians determined by one- and two-point correlations.

$$p_N = A \exp\left[- \sum_{j,k=1}^{N} \alpha_{jk} [y_j - \langle n(t_j) \rangle][y_k - \langle n(t_k) \rangle] \right]$$

$$\boldsymbol{\alpha} = \mathbf{V}^{-1}, \quad V_{jk} = \langle n(t_j) n(t_k) \rangle - \langle n(t_j) \rangle \langle n(t_k) \rangle$$

2. Linear combinations of n at different times are Gaussian random variables.

$$A = \int f(t) n(t) dt \,, \quad B = \int g(t) n(t) dt$$

$$\langle A \rangle = \int f(t) \langle n(t) \rangle dt \,, \quad \langle B \rangle = \int g(t) \langle n(t) \rangle dt$$

$$\langle AB \rangle = \int f(t_1) g(t_2) \langle n(t_1) n(t_2) \rangle dt_1 dt_2$$

Spectral Density

Consider stationary Gaussian random variable, with zero mean.

Stationary: different Fourier components are statistically independent.

$$\tilde{X}(\Omega) \equiv \int_{-\infty}^{+\infty} X(t) e^{i\Omega t} dt$$

$$\langle \tilde{X}(\Omega) \tilde{X}^*(\Omega') \rangle = 2\pi\delta(\Omega - \Omega') S_X(\Omega) \qquad S_X(\Omega) = \int_{-\infty}^{+\infty} C_X(t) e^{i\Omega t} dt$$

$$C_X(t) = \int_{-\infty}^{+\infty} \frac{d\Omega}{2\pi} S_X(\Omega) e^{-i\Omega t}$$

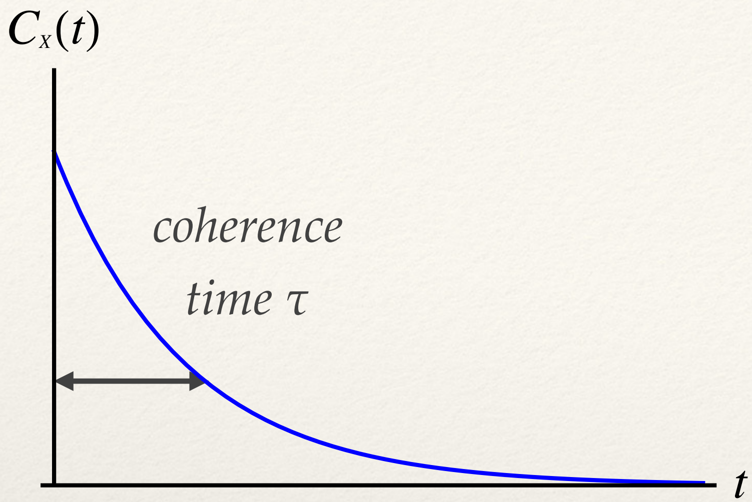
$$\langle X^2(0) \rangle = C(0) = \int_{-\infty}^{+\infty} \frac{d\Omega}{2\pi} S_X(\Omega)$$

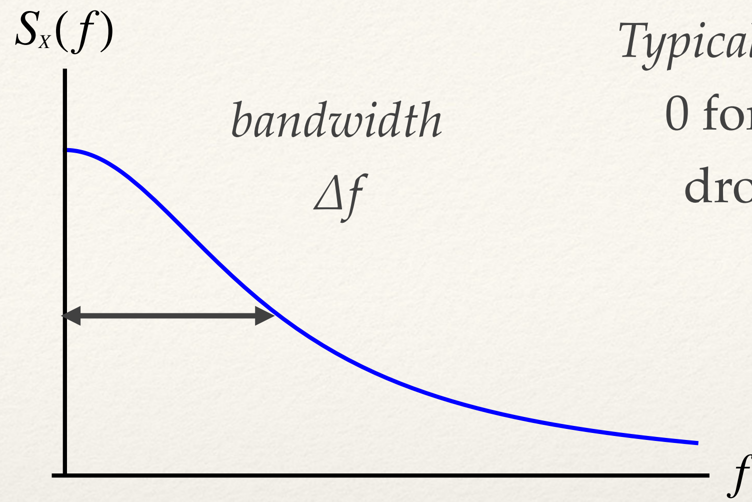
Spectral density characterizes

the amount of noise power per unit frequency band.

Cross Spectral Density: $\langle \tilde{X}(\Omega) \tilde{Y}^*(\Omega') \rangle = 2\pi\delta(\Omega - \Omega') S_{XY}(\Omega)$

Coherence time, Bandwidth, etc.





Typically, C(t) drops to 0 for large t and S drops for large f

$$\Delta f \cdot \tau \sim 1$$

When sampling noise: duration T of each stretch must be long enough to capture low-frequency features; sample interval Δt must be small enough to capture high-frequency features.

White Noise refers to X with S_X=const

$$\langle X(t)X(t')\rangle = S_X\delta(t-t')$$

Same noise power in each unit frequency band. Total variance is infinite.

In practice, bandwidth always cut off by *device* or *sampling rate*.

$$f_{\max} = \frac{2}{\Delta t}$$

Linear Filters

$$\langle \tilde{X}(\Omega)\tilde{X}^*(\Omega')\rangle = 2\pi\delta(\Omega-\Omega')S_X(\Omega)$$

Application of a linear filter modifies spectral density.

$$Y(t) = \int_{-\infty}^{+\infty} K(t-t')X(t')dt'$$

$$\tilde{Y}(\Omega) = \tilde{K}(\Omega)\tilde{X}(\Omega)$$

$$\boxed{S_Y(\Omega) = |\tilde{K}(\Omega)|^2 S_X(\Omega)}$$

Example: output is white noise plus a low-pass filtering of x

$$y = n + \frac{i\gamma}{\Omega + i\gamma}x$$

$$\bar{x} = \frac{\Omega + i\gamma}{i\gamma}y = x + \frac{\Omega + i\gamma}{i\gamma}n$$

Estimator for x involves combining y and its derivative.

$$S_{\bar{x}} = 1 + \left(\frac{\Omega}{\gamma}\right)^2$$

Measuring Spectral Density

$$S_X(\Omega) = \left\langle \lim_{T\to+\infty} \frac{1}{T} \left| \int_{-T/2}^{+T/2} X(t) e^{i\Omega t} dt \right|^2 \right\rangle$$

In practice, *spectral density* can be measured *by averaging over the spectra of finite patches of data*. (T should be much longer than *coherence time*)

Searching for Signal with Known Shape

$$x = n + h$$

data = noise + (possible signal)

n is a stationary Gaussian random process with zero mean.

Let us construct a ***detection statistic***

$$\rho = \int_{-\infty}^{+\infty} x(t)y(t)dt$$

a template

$$\rho = N + S$$

$$N = \int_{-\infty}^{+\infty} n(t)y(t)\,dt\,, \quad S = \int_{-\infty}^{+\infty} h(t)y(t)\,dt\,.$$

Gaussian random variable with zero mean.

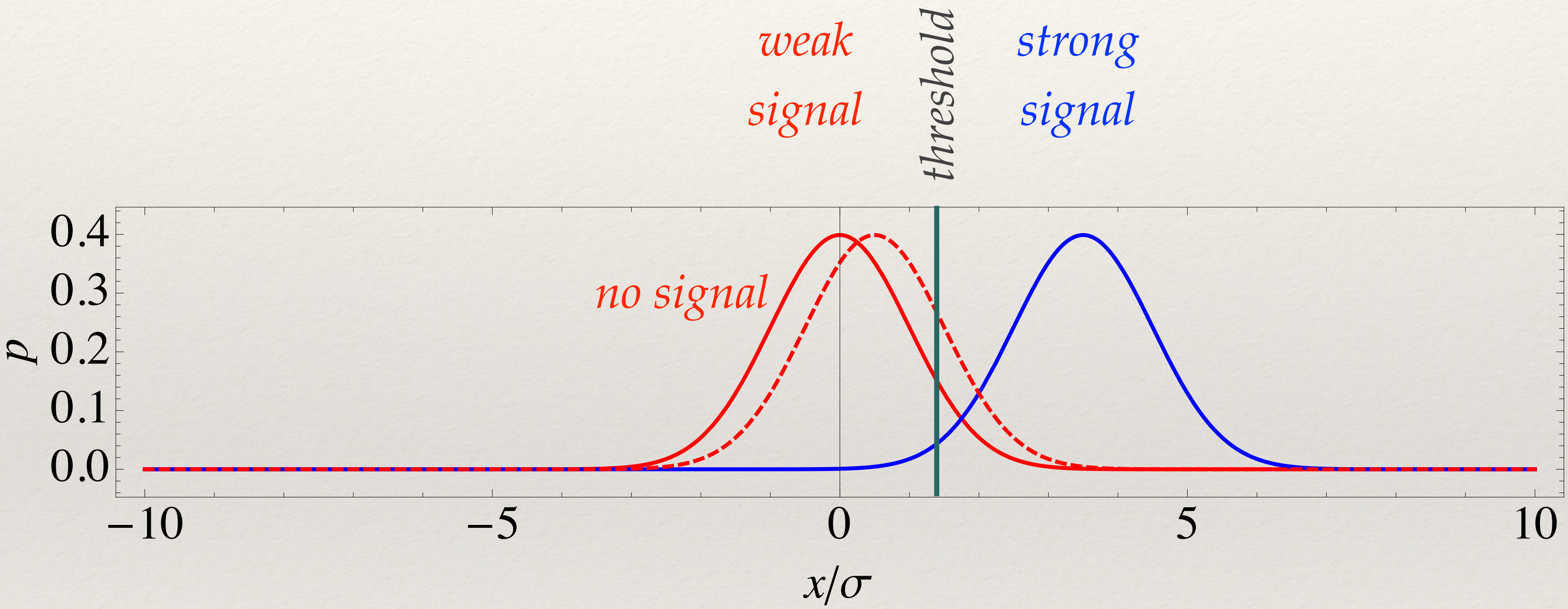
a non-zero constant (when signal present) or zero (when signal absent)

$$p_N(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}} \qquad \sigma^2 \equiv \langle N^2 \rangle = \int_{-\infty}^{+\infty} |\tilde{y}^2(\Omega)| S_n(\Omega) \frac{d\Omega}{2\pi}$$

Searching for Signal with Known Shape

$$\rho = N + S \qquad N = \int_{-\infty}^{+\infty} n(t)y(t)\,dt\,, \quad S = \int_{-\infty}^{+\infty} h(t)y(t)\,dt\,.$$

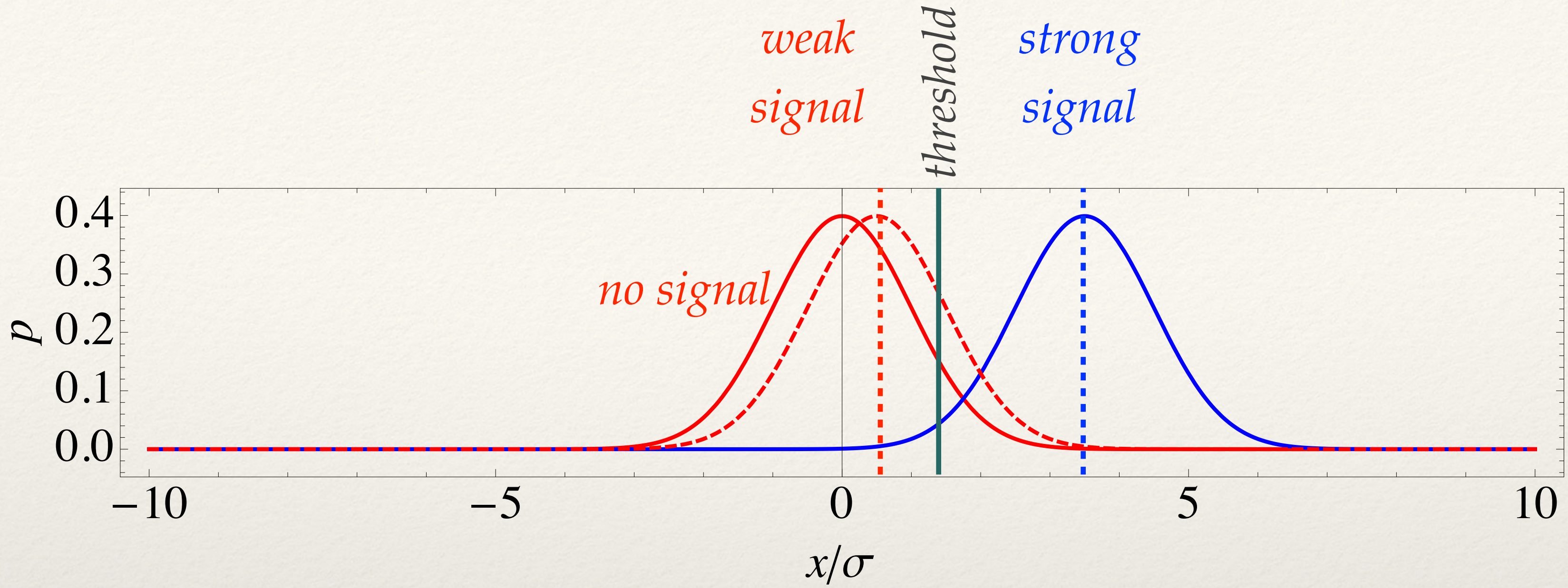
$$p_N(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}} \qquad \sigma^2 \equiv \langle N^2 \rangle = \int_{-\infty}^{+\infty} |\tilde{y}^2(\Omega)| S_n(\Omega) \frac{d\Omega}{2\pi}$$



Set detection threshold of $\rho > \rho_*$

Threshold must be large enough so that false-alarm probability is low.
Signal must be strong enough in order to pass threshold.

Searching for Signal with Known Shape



Set detection threshold of $\rho > \rho_*$ *false-alarm probability:* $\epsilon = \frac{1}{2}\mathrm{erfc}\left(\frac{\rho_*}{\sqrt{2}\sigma}\right)$

In presence of signal: $S > \rho_*$*, more than 50% chance detectable*

$$S_* = \rho_* = \sqrt{2}\sigma\,\mathrm{erfc}^{-1}(2\epsilon)$$

Probability for detection, when signal present. $\frac{1}{2}\mathrm{erfc}\left[\frac{\rho_* - S}{\sqrt{2}\sigma}\right] = \frac{1}{2}\mathrm{erfc}\left[\mathrm{erfc}^{-1}(2\epsilon) - \frac{S}{\sqrt{2}\sigma}\right]$

Optimal Filter

$$x = n + h$$

$$\rho = \int_{-\infty}^{+\infty} x(t)y(t)dt \qquad \rho = N + S$$

$$N = \int_{-\infty}^{+\infty} n(t)y(t)\,dt\,, \quad S = \int_{-\infty}^{+\infty} h(t)y(t)\,dt\,.$$

Set detection threshold of $\rho > \rho_*$ *false-alarm probability:* $\epsilon = \frac{1}{2}\text{erfc}\left(\frac{\rho_*}{\sqrt{2}\sigma}\right)$

$$\sigma^2 \equiv \langle N^2 \rangle = \int_{-\infty}^{+\infty} |\tilde{y}^2(\Omega)| S_n(\Omega) \frac{d\Omega}{2\pi}$$

Probability for detection, when signal present. $\frac{1}{2}\text{erfc}\left[\frac{\rho_* - S}{\sqrt{2}\sigma}\right] = \frac{1}{2}\text{erfc}\left[\text{erfc}^{-1}(2\epsilon) - \frac{S}{\sqrt{2}\sigma}\right]$

need to adjust y(t) to maximize S/σ, ***signal-to-noise ratio*** $\frac{S^2}{\sigma^2} = \frac{\left|\int \tilde{y}^*(\Omega)\tilde{h}(\Omega)\frac{d\Omega}{2\pi}\right|^2}{\int S_n(\Omega)|y^2(\Omega)|\frac{d\Omega}{2\pi}}$

Optimal Filter

$$\frac{S^2}{\sigma^2} = \frac{\left| \int \tilde{y}^*(\Omega)\tilde{h}(\Omega)\frac{d\Omega}{2\pi} \right|^2}{\int S_n(\Omega)|y^2(\Omega)|\frac{d\Omega}{2\pi}}$$

define $\langle A|B\rangle \equiv \int \frac{\tilde{A}^*(\Omega)\tilde{B}(\Omega)}{S_n(\Omega)}\frac{d\Omega}{2\pi}$

the Cauchy inequality gives $\frac{S^2}{\sigma^2} = \frac{|\langle yS_n|h\rangle|^2}{\langle yS_n|yS_n\rangle} \leq \langle h|h\rangle$

equality sign holds only when $\tilde{y}(\Omega) = \frac{\tilde{h}(\Omega)}{S_n(\Omega)}$ *template is signal inversely weighted by spectrum*

$$\left(\frac{S}{\sigma}\right)_{\text{opt}} = \sqrt{\langle h|h\rangle} = \sqrt{\int_{-\infty}^{+\infty} \frac{|\tilde{h}^2(\Omega)|}{S_n(\Omega)}\frac{d\Omega}{2\pi}}$$

Regions with low S are weighted more in the integral.

Fourier Transform of a Chirp Signal

$$h(t) = A(t)e^{-i\Phi(t)}, \quad \Omega_h(t) \equiv \dot{\Phi}(t)$$

A and Ω are slowly varying compared with Ω

The Fourier transform

$$h(\Omega) = \int A(t)e^{-i\Phi(t)}e^{i\Omega t}dt$$

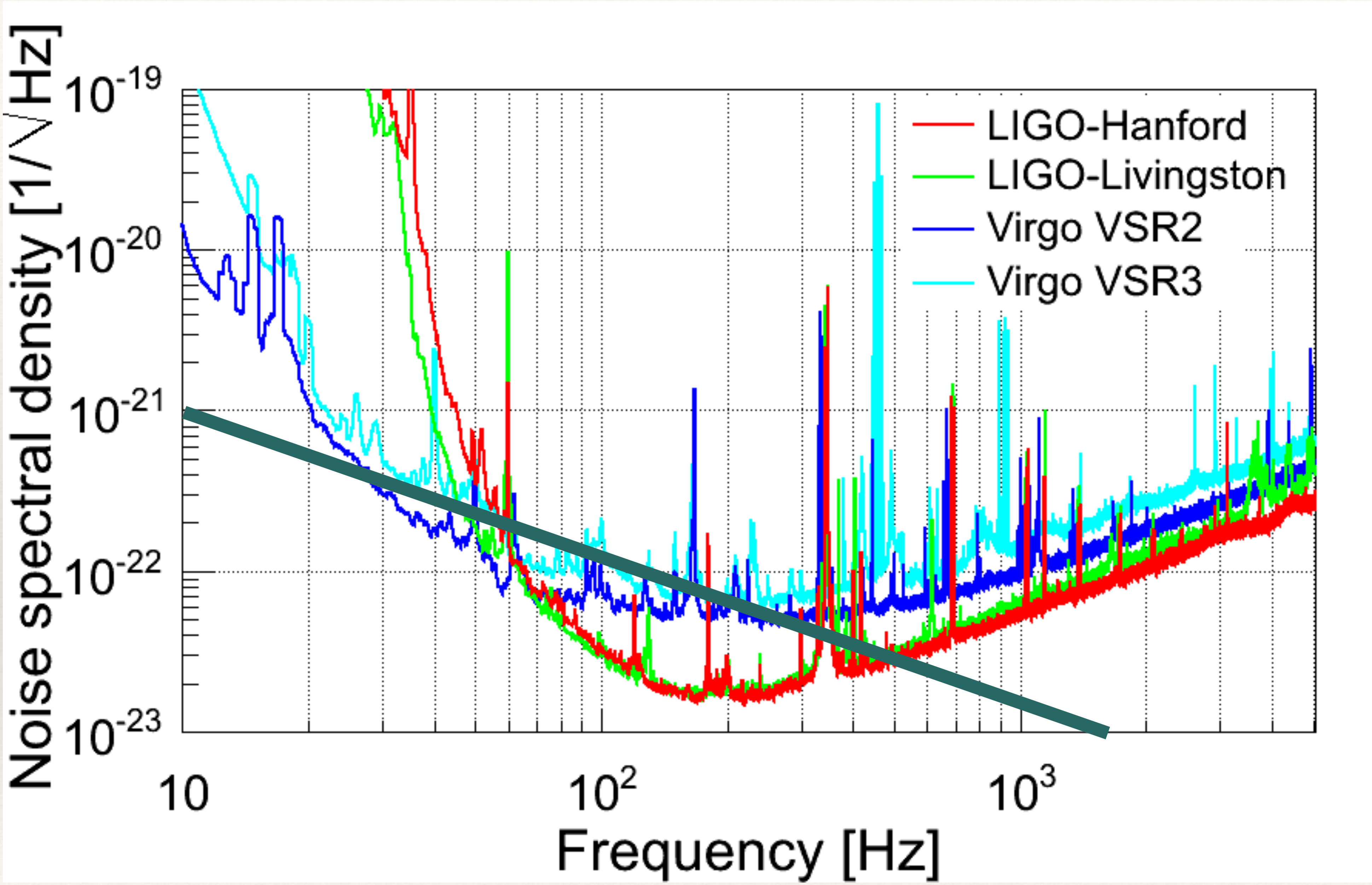
only gets contribution when the phase is stationary, around

$$\Omega \approx \Omega_h(t)$$

$$\tilde{h}(\Omega) = \frac{A(t_\Omega)}{\dot{\Omega}_h(t_\Omega)}e^{-i\Phi(t_\Omega)+i\Omega t_\Omega - \frac{\pi}{4}i}, \quad \Omega_h(t_\Omega) \equiv \Omega$$

for compact binaries

$$|\tilde{h}(\Omega)| \propto \Omega^{-7/6} \qquad \frac{S}{\sigma} \propto \int \frac{\Omega^{-7/3}}{S_n(\Omega)}\frac{d\Omega}{2\pi}$$


Noise spectral density [1/√Hz]
10^-19
10^-20
10^-21
10^-22
10^-23
10
10^2
10^3
Frequency [Hz]
LIGO-Hanford
LIGO-Livingston
Virgo VSR2
Virgo VSR3

Searching for signals with unknown arrival time and phase

$$\rho(t_0) = \int_{-\infty}^{+\infty} \frac{\tilde{h}^*(\Omega) e^{i\Omega t_0} \tilde{x}(\Omega)}{S_n(\Omega)} \frac{d\Omega}{2\pi}$$

signal filtered using templates with different arrival times

also an inverse Fourier transform

(can be done numerically with FFT, much faster than doing integrals for each t_0.)

$$h(\Omega) = h_0(\Omega) e^{i\phi \cdot \mathrm{sgn}(\Omega)}$$

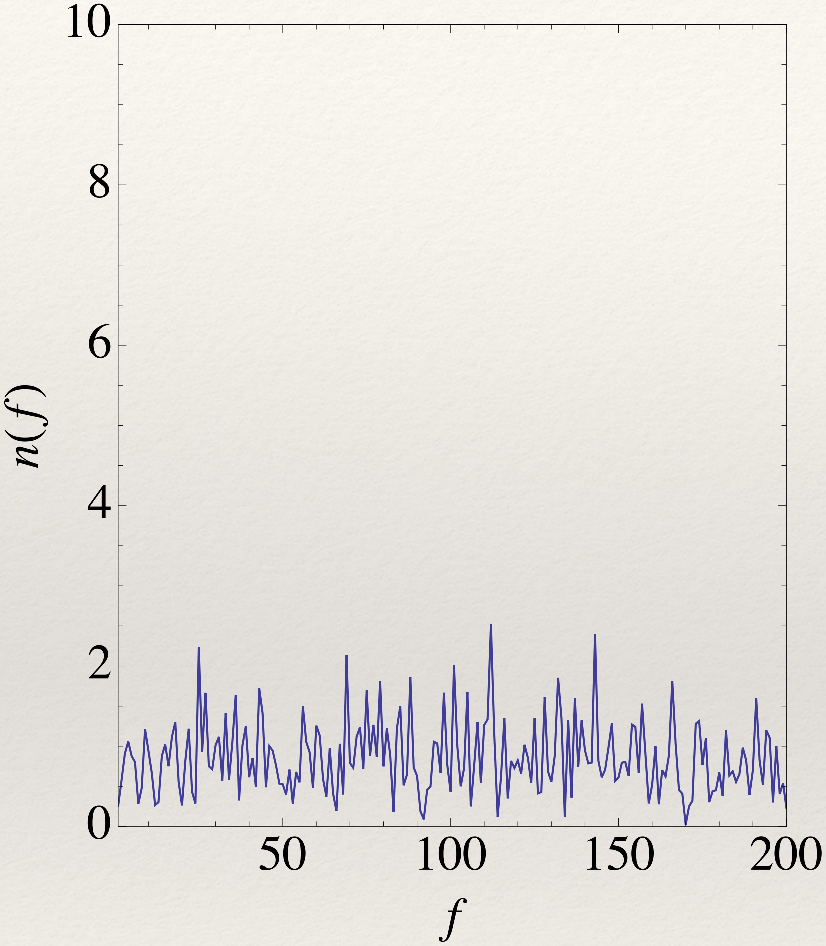
$$\rho(\phi, t_0) = 2\mathrm{Re}\left[\int_0^{+\infty} \frac{\left[\tilde{h}_0^*(\Omega) e^{-i\phi + i\Omega t_0}\right] \tilde{x}(\Omega)}{S_n(\Omega)} \frac{d\Omega}{2\pi}\right]$$

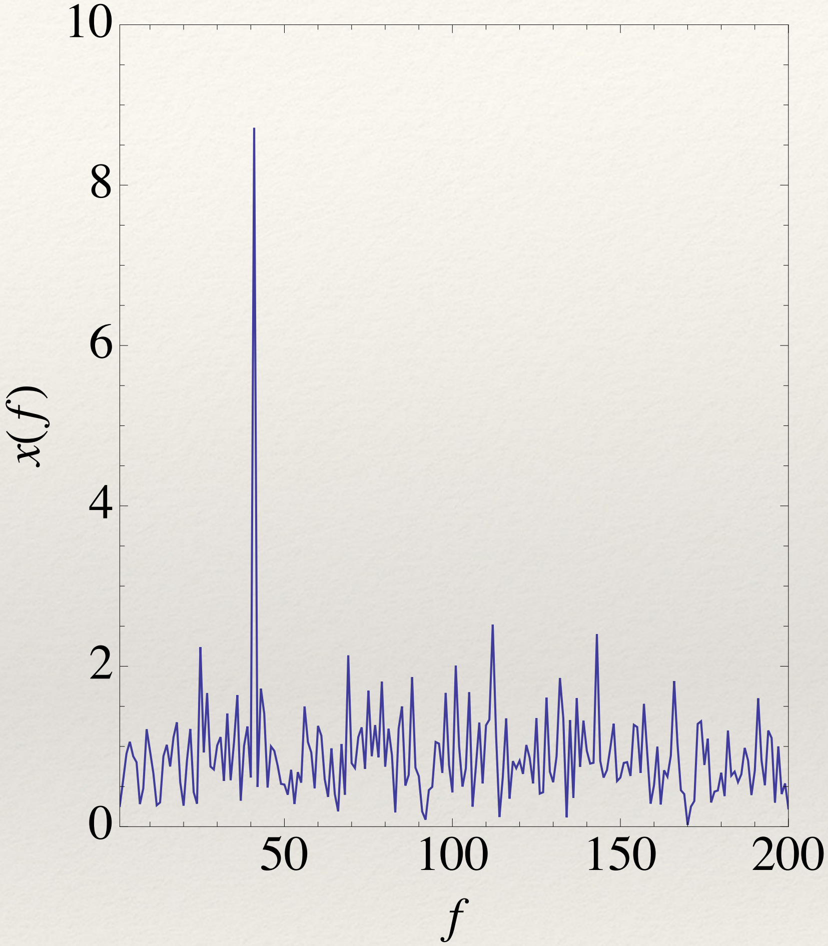
$$\max_{\phi} \rho(\phi, t_0) = 2\left|\int_0^{+\infty} \frac{\left[\tilde{h}_0^*(\Omega) e^{i\Omega t_0}\right] \tilde{x}(\Omega)}{S_n(\Omega)} \frac{d\Omega}{2\pi}\right|$$

No need to search over different values of ϕ

As multiple templates are used, threshold must be reset. to keep the same false-alarm probability.

Searching for signals with unknown frequency





Summary

- Stationary (Gaussian) Noises can be described (fully) by their spectral densities.
- Spectral Density characterizes noise power per unit frequency band.
- Signal with known shape should be extracted from noisy data with an optimal filter.

Projects

1. Gravitational waveforms from compact binaries.
    A. Write a code generates gravitational waveforms from binaries in **circular orbits.** [Here we need to express time and distance in physical units.]
    B. Fourier transform this waveform *numerically*, and compare with the "Stationary-Phase Approximation".
2. Generating Gaussian Noise.
    A. Write a code that can generate "white noise". Filter your white noise, and measure the noise spectrum of the output of the filters.
    B. Find the "Advanced LIGO noise spectrum", and approximate it by sections of power laws.
    C. Generate Advanced LIGO noise.
3. Matched filtering.
    A. Add the compact-binary waveform onto Advanced LIGO noise, and use a matched filter to search for this waveform.