

Class Preparation

* bring laptop if you own one

* download programs prior to class
→ Piazza

* stay up to date on piazza

* New to Matlab? Work through

intro1.m
intro2.m } gravitational waves
intro3.m } from Black holes

4 credit class: take more preparation
before class

Linear interpolation

- Assume that the function $f(x)$ in the interval (x_1, x_2) is such that $f(x_1)$ and $f(x_2)$ have opposite signs.
- Introduce the straight line function $y(x)$ connecting the points $(x_1, f(x_1))$ and $(x_2, f(x_2))$:

$$y(x) = f(x_1) + \frac{f(x_1) - f(x_2)}{x_1 - x_2} (x - x_1)$$

- Denote by x_3 the position at which $y(x)$ crosses the axis Ox . This point is given by

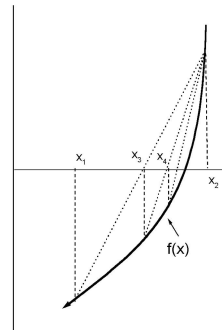
$$x_3 = \frac{x_2 f(x_1) - x_1 f(x_2)}{f(x_1) - f(x_2)} = x_2 - \frac{f(x_2)}{f(x_2) - f(x_1)} (x_2 - x_1)$$

- If in the (x_1, x_2) interval, $f(x) \approx y(x)$, then x_3 is a **good** approximation to the root r of $f(x)$.

Linear interpolation: Recurrence relation

With x_3 , there are three choices:

- ① If $f(x_1) \cdot f(x_3) < 0$ set $x_2 = x_3$
- ② If $f(x_2) \cdot f(x_3) < 0$ set $x_1 = x_3$
- ③ If $f(x_3) = 0$ root has been found



Recurrence relation:

$$x_{n+2} = x_{n+1} - \frac{f(x_{n+1})}{f(x_{n+1}) - f(x_n)} (x_{n+1} - x_n)$$

Linear interpolation

```
REPEAT
  SET  $x_3 = x_2 - f(x_2) \cdot \frac{x_2 - x_1}{f(x_2) - f(x_1)}$ 
  IF  $f(x_3) \cdot f(x_1) < 0$ 
    SET  $x_2 = x_3$ 
  ELSE
    SET  $x_1 = x_3$ 
  ENDIF
UNTIL  $|f(x_3)| < E$ 
```

Linear interpolation : Convergence

THEOREM: If ξ is the root of the equation $f(x) = 0$, and $\varepsilon_n = x_n - \xi$ is the error associated with the root estimate x_n , the convergence rate of the linear interpolation method is

$$\varepsilon_{n+1} = k \cdot \varepsilon_n^{1.618}$$

PROOF: From $x_n = \xi + \varepsilon_n$, we have that

$$\begin{aligned} f(x_n) &= f(\xi + \varepsilon_n) = f(\xi) + \varepsilon_n f'(\xi) + \frac{\varepsilon_n^2}{2} f''(\xi) \\ &= \varepsilon_n f'(\xi) + \frac{\varepsilon_n^2}{2} f''(\xi) \end{aligned}$$

then from the recurrence relation

$$x_{n+2} = x_{n+1} - \frac{f(x_{n+1})}{f(x_{n+1}) - f(x_n)} (x_{n+1} - x_n)$$

one has that

$$\varepsilon_{n+2} = \varepsilon_{n+1} - \frac{\varepsilon_{n+1} f'(\xi) + \frac{1}{2} \varepsilon_{n+1}^2 f''(\xi)}{f'(\xi)(\varepsilon_{n+1} - \varepsilon_n) + \frac{1}{2} f''(\xi)(\varepsilon_{n+1}^2 - \varepsilon_n^2)} \cdot (\varepsilon_{n+1} - \varepsilon_n)$$

$$\varepsilon_{n+2} = \varepsilon_{n+1} - \frac{\varepsilon_{n+1} f'(\xi) + \frac{1}{2} \varepsilon_{n+1}^2 f''(\xi)}{f'(\xi) + \frac{1}{2} f''(\xi)(\varepsilon_{n+1} + \varepsilon_n)}$$

$$\varepsilon_{n+2} = \varepsilon_{n+1} \left[1 - \frac{f'(\xi) + \frac{1}{2} \varepsilon_{n+1} f''(\xi)}{f'(\xi) + \frac{1}{2} f''(\xi)(\varepsilon_{n+1} + \varepsilon_n)} \right]$$

$$\varepsilon_{n+2} = \varepsilon_{n+1} \left[\frac{\frac{1}{2} \varepsilon_n f''(\xi)}{f'(\xi) + \frac{1}{2} f''(\xi)(\varepsilon_{n+1} + \varepsilon_n)} \right]$$

$$\varepsilon_{n+2} \approx \varepsilon_{n+1} \varepsilon_n \frac{1}{2} \frac{f''(\xi)}{f'(\xi)}$$

Let $\varepsilon_{n+1} = k \varepsilon_n^m$, then

$$\begin{aligned}\varepsilon_{n+2} &= k \varepsilon_{n+1}^m = k^{m+1} \varepsilon_n^{m^2} \\ \varepsilon_{n+1} \varepsilon_n &= k \varepsilon_n^{m+1}\end{aligned}$$

thus

$$\varepsilon_{n+2} \approx \varepsilon_{n+1} \varepsilon_n \frac{1}{2} \frac{f''(\xi)}{f'(\xi)}$$

implies

$$k^{m+1} \varepsilon_n^{m^2} = k \varepsilon_n^{m+1} \frac{f''(\xi)}{2f'(\xi)}$$

Therefore

$$\begin{aligned}k^m &= \frac{f''(\xi)}{2f'(\xi)} \\ m^2 &= m + 1 \Rightarrow m = 1.618\end{aligned}$$

$$\varepsilon_{n+1} = k \cdot \varepsilon_n^{1.618}$$

Newton Raphson method

If in the neighborhood of the root r of the equation $f(x) = 0$ the 1st and 2nd derivatives of $f(x)$ are continuous, it is possible to develop a root finding method which is faster than the bisection and linear interpolation method.

Let x_{n+1} and x_n be respectively the $n + 1$ and n root iterations of the equation $f(x) = 0$ such that $x_{n+1} = x_n + \delta_n$ with $\delta_n \ll 1$.

Then:

$$\begin{aligned} f(x_{n+1}) &= f(x_n + \delta_n) \\ &= f(x_n) + \delta_n f'(x_n) + \frac{\delta_n^2}{2} f''(x_n) + \dots \end{aligned}$$

In the limit $n \rightarrow \infty$, we have that $f(x_{n+1}) = 0$, thus

$$0 = f(x_n) + \delta_n f'(x_n) \quad \Rightarrow \quad \delta_n = -\frac{f(x_n)}{f'(x_n)}$$

Newton Raphson method: Recurrence relation

Then

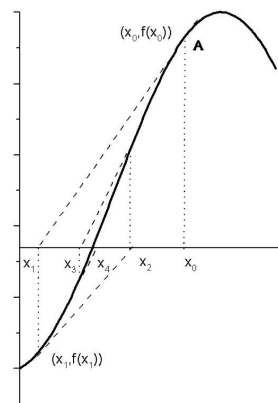
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Notice problems when

- $f'(x_n) = 0$
- $x_{n+2} = x_n$ non-convergent cycle

Recall linear interpolation recurrence relation

$$x_{n+1} = x_n - f(x_n) \left[\frac{(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})} \right]$$



Newton Raphson method : Convergence

If $f(r) = 0$ such that $x_n = r + \varepsilon_n$, from the recurrence relation

$$\begin{aligned}
 r + \varepsilon_{n+1} &= r + \varepsilon_n - \frac{f(r + \varepsilon_n)}{f'(r + \varepsilon_n)} \\
 \varepsilon_{n+1} &= \varepsilon_n - \frac{f(r) + \varepsilon_n f'(r) + \frac{1}{2} \varepsilon_n^2 f''(r)}{f'(r) + \varepsilon_n f''(r)} \\
 \varepsilon_{n+1} &= \varepsilon_n - \frac{\varepsilon_n f'(r) + \frac{1}{2} \varepsilon_n^2 f''(r)}{f'(r) + \varepsilon_n f''(r)} \\
 \frac{\varepsilon_{n+1}}{\varepsilon_n} &= 1 - \frac{f'(r) + \frac{1}{2} \varepsilon_n f''(r)}{f'(r) + \varepsilon_n f''(r)} \\
 \frac{\varepsilon_{n+1}}{\varepsilon_n} &= \frac{\frac{1}{2} \varepsilon_n f''(r)}{f'(r) + \varepsilon_n f''(r)} \approx \varepsilon_n \frac{1}{2} \frac{f''(r)}{f'(r)}
 \end{aligned}$$

we get that

$$\varepsilon_{n+1} = -\frac{f''(r)}{2f'(r)} \cdot \varepsilon_n^2$$

That is, quadratic convergence

Newton-Raphson

```
IF  $f'(x) \neq 0$   
  REPEAT  
    SET  $x = x - f(x)/f'(x)$   
  UNTIL  $|f(x)| < E$   
ENDIF
```

Newton Halley method

Recall

$$f(x_{n+1}) = f(x_n + \delta_n) = f(x_n) + \delta_n f'(x_n) + \frac{\delta_n^2}{2} f''(x_n) + \dots$$

$$f(x_n) + \delta_n \left[f'(x_n) + \frac{\delta_n}{2} f''(x_n) \right] = 0$$

$$\delta_n = - \frac{f(x_n)}{f'(x_n) + \frac{\delta_n}{2} f''(x_n)}$$

Substitute the previous Newton Raphson result

$$\delta_n \approx - \frac{f(x_n)}{f'(x_n)}$$

in the denominator

Your Work

- * add the missing lines to findroot.m
- * run code
- * study differences between the methods

Then

$$\delta_n = -\frac{f(x_n)}{f'(x_n) - \frac{f(x_n)f''(x_n)}{2f'(x_n)}}$$

so

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n) - \frac{f(x_n)f''(x_n)}{2f'(x_n)}}$$

$$x_{n+1} = x_n - \frac{2f(x_n)f'(x_n)}{2f'(x_n)f'(x_n) - f(x_n)f''(x_n)}$$

Notice that for $f''(x_n) = 0$ we recover that traditional Newton-Raphson method.

Newton Halley's methods achieves **cubic** convergence:

$$\varepsilon_{n+1} = -\left[\frac{1}{6} \frac{f'''(\xi)}{f'(\xi)} - \frac{1}{4} \left(\frac{f''(\xi)}{f'(\xi)}\right)^2\right] \cdot \varepsilon_n^3$$

Newton-Halley

```
IF  $f'(x) \neq 0$  AND  $f''(x) \neq 0$ 
  REPEAT
    SET  $x = x - 2 f(x) f'(x) / (2 f'(x) f'(x) - f(x) f''(x))$ 
  UNTIL  $|f(x)| < E$ 
ENDIF
```

EXAMPLE: Use the Newton-Raphson and Newton-Halley method to calculate square root of a number Q (here $Q = 9$). Set $f(x) = x^2 - Q$, then

$$x_{n+1} = \frac{1}{2} \left(x_n + \frac{Q}{x_n} \right) \quad (\text{Raphson}) \quad x_{n+1} = \frac{x_n^3 + 3 x_n Q}{3 x_n^2 + Q} \quad (\text{Halley})$$

Newton	Error	Halley	Error
$x_0=15$	$\varepsilon_0 = 12$	$x_0=15$	$\varepsilon_0 = 12$
$x_1=7.8$	$\varepsilon_1 = 4.8$	$x_1=5.526$	$\varepsilon_1 = 2.5$
$x_2=4.477$	$\varepsilon_2 = 1.477$	$x_2=3.16024$	$\varepsilon_2 = 0.16$
$x_3=3.2436$	$\varepsilon_3 = 0.243$	$x_3=3.00011$	$\varepsilon_3 = 1.05 \times 10^{-4}$
$x_4=3.0092$	$\varepsilon_4 = 9.15 \times 10^{-3}$	$x_4=3.0000...$	$\varepsilon_4 = 3.24 \times 10^{-14}$

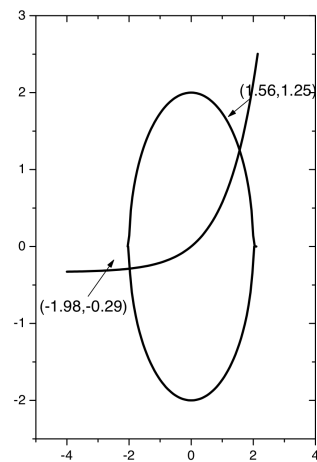
Non-linear systems of equations

An example of two non-linear equations is the following:

$$f(x, y) = e^x - 3y - 1$$

$$g(x, y) = x^2 + y^2 - 4$$

$f(x, y) = 0$ and $g(x, y) = 0$ are two curves on the xy plane.



Newton Raphson method for 2 equations

Let's assume that after $n + 1$ iterations the method converged to the solution (x_{n+1}, y_{n+1}) i.e. $f(x_{n+1}, y_{n+1}) \approx 0$ and $g(x_{n+1}, y_{n+1}) \approx 0$. Then if $x_{n+1} = x_n + \varepsilon_n$ and $y_{n+1} = y_n + \delta_n$, then

$$0 \approx f(x_{n+1}, y_{n+1}) = f(x_n + \varepsilon_n, y_n + \delta_n) \approx f(x_n, y_n) + \varepsilon_n \left(\frac{\partial f}{\partial x} \right)_n + \delta_n \left(\frac{\partial f}{\partial y} \right)_n$$

$$0 \approx g(x_{n+1}, y_{n+1}) = g(x_n + \varepsilon_n, y_n + \delta_n) \approx g(x_n, y_n) + \varepsilon_n \left(\frac{\partial g}{\partial x} \right)_n + \delta_n \left(\frac{\partial g}{\partial y} \right)_n$$

Solving for ε_n and δ_n

$$\varepsilon_n = \frac{-f \frac{\partial g}{\partial y} + g \frac{\partial f}{\partial y}}{\frac{\partial f}{\partial x} \frac{\partial g}{\partial y} - \frac{\partial g}{\partial x} \frac{\partial f}{\partial y}} \quad \text{and} \quad \delta_n = \frac{-g \frac{\partial f}{\partial x} + f \frac{\partial g}{\partial x}}{\frac{\partial f}{\partial x} \frac{\partial g}{\partial y} - \frac{\partial g}{\partial x} \frac{\partial f}{\partial y}}$$

Since $x_{n+1} = x_n + \varepsilon_n$ and $y_{n+1} = y_n + \delta_n$, the recurrence relations are:

$$\begin{aligned}x_{n+1} &= x_n - \left(\frac{f \cdot g_y - g \cdot f_y}{f_x \cdot g_y - g_x \cdot f_y} \right)_n \\y_{n+1} &= y_n - \left(\frac{g \cdot f_x - f \cdot g_x}{f_x \cdot g_y - g_x \cdot f_y} \right)_n\end{aligned}$$

where $f_x = \frac{\partial f}{\partial x}$