# Computational Physics and Astrophysics

## Partial Differential Equations

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# Introduction

- A differential equation involving more than one independent variable is called a partial differential equation (PDE)
- Many problems in applied science, physics and engineering are modeled mathematically with PDE.
- We will mostly focus on finite-difference methods to solve numerically PDEs.
- PDEs are classified as one of three types, with terminology borrowed from the conic sections.
- That is, for a 2nd-degree polynomial in $x$ and $y$

$$Ax^2 + Bxy + Cy^2 + D = 0$$

the graph is a quadratic curve, and when

  - $B^2 - 4AC < 0$ the curve is a ellipse,
  - $B^2 - 4AC = 0$ the curve is a parabola
  - $B^2 - 4AC > 0$ the curve is a hyperbola

Similarly, given

$$A\frac{\partial^2\psi}{\partial x^2} + B\frac{\partial^2\psi}{\partial x\partial y} + C\frac{\partial^2\psi}{\partial y^2} + D\left(x, y, \psi, \frac{\partial\psi}{\partial x}, \frac{\partial\psi}{\partial x}\right) = 0$$

where $A$, $B$ and $C$ are constants. There are 3 types of equations:

- If $B^2 - 4AC < 0$, the equation is called elliptic,
- If $B^2 - 4AC = 0$, the equation is called parabolic
- If $B^2 - 4AC > 0$, the equation is called hyperbolic

The classification can be extended to PDEs in more than two dimensions.

Two classic examples of elliptic PDEs are the Laplace and Poisson equations:

$$\nabla^2 \phi = 0 \quad \text{and} \quad \nabla^2 \phi = \rho$$

where in 3D

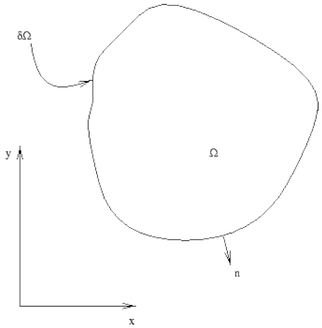
$$\begin{aligned}
\nabla^2 &= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \\
\nabla^2 &= \frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \frac{1}{r^2 \sin\theta}\frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2} \\
\nabla^2 &= \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\sin\theta\frac{\partial}{\partial \theta} + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2}{\partial \phi^2}
\end{aligned}$$

Boundary-value Problem

$$\nabla^2 \phi = \rho \quad \text{in a domain } \Omega$$

Boundary Conditions

- Dirichlet: $\phi = b_1$ on $\partial\Omega$
- Neumann: $\frac{\partial \phi}{\partial n} = \hat{n} \cdot \nabla \phi = b_2$ on $\partial\Omega$
- Robin: $\frac{\partial \phi}{\partial n} + \alpha\, \phi = \hat{n} \cdot \nabla \phi = b_3$ on $\partial\Omega$



Classic examples of hyperbolic PDEs are:

$$-\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} + \nabla^2\psi = 0 \quad \text{wave equation}$$

$$\frac{\partial\psi}{\partial t} + \vec{V}\cdot\nabla\psi = 0 \quad \text{advection equation}$$

Classic example of parabolic PDEs are

$$\frac{\partial\psi}{\partial t} - \nabla\cdot(D\,\nabla\psi) = 0 \quad \text{diffusion equation}$$

$$\frac{\partial\psi}{\partial t} - \alpha\,\nabla^2\psi = 0 \quad \text{heat equation}$$

Notice from the diffusion equation that

$$\frac{\partial\psi}{\partial t} + \nabla\cdot\vec{J} = 0 \quad \text{continuity equation}$$

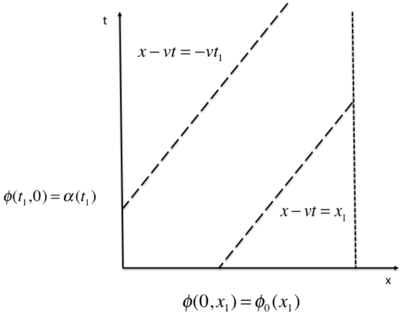
$$\vec{J} = -D\,\nabla\psi \quad \text{Fick's first law}$$

# Advection or Convection Equation

- Let's consider the 1D case

$$\partial_t \phi + v\,\partial_x \phi = 0$$

  with $v = \text{const} > 0$, $t \geq 0$ and $x \in [0, 1]$
- Initial data: $\phi(0, x) = \phi_0(x)$
- Boundary conditions: $\phi(t, 0) = \alpha(t)$ and $\phi(t, 1) = \beta(t)$
- Solutions to this equation have the form $\phi(t, x) = \phi(x - v\,t)$
- Therefore, the solution $\phi(t, x)$ is constant along the lines $x - v\,t = \text{const}$ called characteristics



# Forward-Time Center-Space (FTCS) Discretization

- Let's consider the following discretization of the differential operators

$$\begin{aligned} \partial_t \phi_i^n &= \frac{\phi_i^{n+1} - \phi_i^n}{\Delta t} + O(\Delta t) \\ \partial_x \phi_i^n &= \frac{\phi_{i+1}^n - \phi_{i-1}^n}{2\,\Delta x} + O(\Delta x^2) \end{aligned}$$

  where we have used the notation $\phi_i^n \equiv \phi(t^n, x_i)$
- Therefore, the finite difference approximation to $\partial_t \phi + v\,\partial_x \phi = 0$ is

$$\frac{(\bar{\phi}_i^{n+1} - \bar{\phi}_i^n)}{\Delta t} + v\,\frac{(\bar{\phi}_{i+1}^n - \bar{\phi}_{i-1}^n)}{2\,\Delta x} = 0$$

- Notice that we are making a distinction between the solution $\phi(t, x)$ to the continuum equation and $\bar{\phi}_i^n$ the solution to the discrete equation.

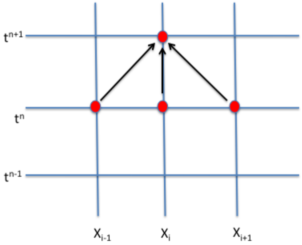
Solving

$$\frac{(\bar{\phi}_i^{n+1} - \bar{\phi}_i^n)}{\Delta t} + v\,\frac{(\bar{\phi}_{i+1}^n - \bar{\phi}_{i-1}^n)}{2\,\Delta x} = 0$$

for $\bar{\phi}_i^{n+1}$, one gets the following relationship to update the solution

$$\bar{\phi}_i^{n+1} = \bar{\phi}_i^n - \frac{1}{2} C \left(\bar{\phi}_{i+1}^n - \bar{\phi}_{i-1}^n\right)$$

where $C \equiv \Delta t\, v / \Delta x$



# Stability

- The tendency for any perturbation in the numerical solution to decay.
- That is, given a discretization scheme, we need to evaluate the degree to which errors introduced at any stage of the computation will grow or decay.
- We are then concerned with the behavior of the solution error

$$\epsilon_i^n = \phi_i^n - \bar{\phi}_i^n$$

- Substitution of $\bar{\phi}_i^n = \phi_i^n - \epsilon_i^n$ into

$$\bar{\phi}_i^{n+1} = \bar{\phi}_i^n - \frac{1}{2} C \left( \bar{\phi}_{i+1}^n - \bar{\phi}_{i-1}^n \right)$$

yields

$$\phi_i^{n+1} - \epsilon_i^{n+1} = \phi_i^n - \epsilon_i^n - \frac{1}{2} C \left( \phi_{i+1}^n - \epsilon_{i+1}^n - \phi_{i-1}^n + \epsilon_{i-1}^n \right)$$

- Substitute the following Taylor expansions

$$\begin{aligned}\phi_i^{n+1} &= \phi_i^n + \Delta t\,\partial_t\phi_i^n + O(\Delta t^2)\\ \phi_{i\pm1}^n &= \phi_i^n \pm \Delta x\,\partial_x\phi_i^n + O(\Delta x^2)\end{aligned}$$

- Then

$$\phi_i^n + \Delta t\,\partial_t\phi_i^n - \epsilon_i^{n+1} = \phi_i^n - \epsilon_i^n$$
$$-\frac{1}{2}C\,\left(\phi_i^n + \Delta x\,\partial_x\phi_i^n - \epsilon_{i+1}^n - \phi_i^n + \Delta x\,\partial_x\phi_i^n + \epsilon_{i-1}^n\right)$$

or

$$\Delta t\,\partial_t\phi_i^n - \epsilon_i^{n+1} = -\epsilon_i^n - \frac{1}{2}C\,\left(2\,\Delta x\,\partial_x\phi_i^n - \epsilon_{i+1}^n + \epsilon_{i-1}^n\right)$$
$$\epsilon_i^{n+1} = \epsilon_i^n - \frac{1}{2}C\,\left(\epsilon_{i+1}^n - \epsilon_{i-1}^n\right) + \Delta t\,\partial_t\phi_i^n - C\,\Delta x\,\partial_x\phi_i^n$$
$$\epsilon_i^{n+1} = \epsilon_i^n - \frac{1}{2}C\,\left(\epsilon_{i+1}^n - \epsilon_{i-1}^n\right) + \Delta t\,(\partial_t\phi_i^n - v\,\partial_x\phi_i^n)$$
$$\epsilon_i^{n+1} = \epsilon_i^n - \frac{1}{2}C\,\left(\epsilon_{i+1}^n - \epsilon_{i-1}^n\right)$$

- That is, the solution error satisfies also the discrete finite differences approximation

$$\epsilon_i^{n+1} = \epsilon_i^n - \frac{1}{2} C \left( \epsilon_{i+1}^n - \epsilon_{i-1}^n \right)$$

- Von Neumann stability analysis: Assume that the errors satisfy a "separation-of-variables" of the form

$$\epsilon_i^n = \xi^n \, e^{I \, x_j} = \xi^n \, e^{I \, k \, \Delta x \, j}$$

  where $I = \sqrt{-1}$, $k = 2\,\pi/\lambda$ and $\xi$ is a complex amplitude. The $n$ in $\xi^n$ is understood to be a power.

- The condition of stability is $|\xi| \leq 1$ for all $k$.

Substitution of $\epsilon_i^n = \xi^n e^{I k \Delta x i}$ into the finite difference equation yields

$$\xi^{n+1} e^{I k \Delta x i} = \xi^n e^{I k \Delta x i} - \frac{1}{2} C \left( \xi^n e^{I k \Delta x (i+1)} - \xi^n e^{I k \Delta x (i-1)} \right)$$

$$\xi^{n+1} = \xi^n - \frac{1}{2} C \left( \xi^n e^{I k \Delta x} - \xi^n e^{-I k \Delta x} \right)$$

$$\xi = 1 - \frac{1}{2} C \left( e^{I k \Delta x} - e^{-I k \Delta x} \right)$$

$$\xi = 1 - \frac{1}{2} C \, 2 \, I \sin (k \Delta x)$$

$$\xi = 1 - I \, C \sin (k \Delta x)$$

$$|\xi|^2 = 1 + C^2 \sin^2 (k \Delta x)$$

Therefore FTCS discretization applied to the advection equation is unstable.

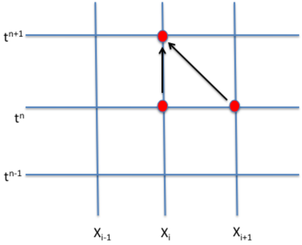
Approximate the advection equation as

$$\frac{(\bar{\phi}_i^{n+1} - \bar{\phi}_i^n)}{\Delta t} + v\,\frac{(\bar{\phi}_{i+1}^n - \bar{\phi}_i^n)}{\Delta x} = 0$$

thus

$$\bar{\phi}_i^{n+1} = \bar{\phi}_i^n - C\,\left(\bar{\phi}_{i+1}^n - \bar{\phi}_i^n\right)$$

where $C \equiv \Delta t\, v / \Delta x$



# FTFS Stability

Substitute $\epsilon_i^n = \xi^n\, e^{I\,k\,\Delta x\,i}$ into

$$\epsilon_i^{n+1} = (1+C)\epsilon_i^n - C\,\epsilon_{i+1}^n$$
$$\xi^{n+1}\, e^{I\,k\,\Delta x\,i} = (1+C)\xi^n\, e^{I\,k\,\Delta x\,i} - C\,\xi^n\, e^{I\,k\,\Delta x\,(i+1)}$$
$$\xi^{n+1} = (1+C)\xi^n - C\,\xi^n\, e^{I\,k\,\Delta x}$$
$$\xi = (1+C) - C\, e^{I\,k\,\Delta x}$$
$$|\xi|^2 = \left[(1+C) - C\, e^{I\,k\,\Delta x}\right]\left[(1+C) - C\, e^{-I\,k\,\Delta x}\right]$$
$$|\xi|^2 = (1+C)^2 + C^2 - (1+C)C\,(e^{I\,k\,\Delta x} + e^{-I\,k\,\Delta x})$$
$$|\xi|^2 = (1+C)^2 + C^2 - 2\,(1+C)C\,\cos(k\,\Delta x)$$
$$|\xi|^2 = 1 + 2\,(1+C)C\,[1-\cos(k\,\Delta x)] \geq 1$$

the method is unstable

# Forward-Time Backward-Space (FTFS) Discretization

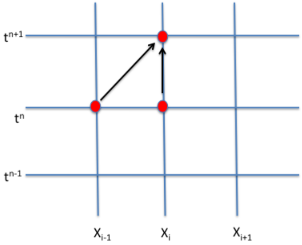
Approximate the advection equation as

$$\frac{(\bar{\phi}_i^{n+1} - \bar{\phi}_i^n)}{\Delta t} + v \frac{(\bar{\phi}_i^n - \bar{\phi}_{i-1}^n)}{\Delta x} = 0$$

thus

$$\bar{\phi}_i^{n+1} = \bar{\phi}_i^n - C \left(\bar{\phi}_i^n - \bar{\phi}_{i-1}^n\right)$$

where $C \equiv \Delta t \, v / \Delta x$



# FTBS Stability

Substitute $\epsilon_i^n = \xi^n \, e^{I\,k\,\Delta x\, i}$ into

$$\epsilon_i^{n+1} = (1 - C)\epsilon_i^n + C\,\epsilon_{i-1}^n$$

$$\xi^{n+1}\,e^{I\,k\,\Delta x\,i} = (1 - C)\xi^n\,e^{I\,k\,\Delta x\,i} + C\,\xi^n\,e^{I\,k\,\Delta x\,(i-1)}$$

$$\xi^{n+1} = (1 - C)\xi^n + C\,\xi^n\,e^{-I\,k\,\Delta x}$$

$$\xi = (1 - C) + C\,e^{-I\,k\,\Delta x}$$

$$|\xi|^2 = \left[(1 - C) + C\,e^{-I\,k\,\Delta x}\right]\left[(1 - C) + C\,e^{I\,k\,\Delta x}\right]$$

$$|\xi|^2 = (1 - C)^2 + C^2 + (1 - C)C\,(e^{I\,k\,\Delta x} + e^{-I\,k\,\Delta x})$$

$$|\xi|^2 = (1 - C)^2 + C^2 + 2\,(1 - C)C\,\cos(k\,\Delta x)$$

$$|\xi|^2 = 1 - 2\,(1 - C)C\,[1 - \cos(k\,\Delta x)]$$

Given

$$\xi|^2 = 1 - 2\,(1-C)C\,[1-\cos(k\,\Delta x)]$$

in order to have $|\xi|^2 \leq 1$

$$-1 \leq 1 - 2\,(1-C)C\,[1-\cos(k\,\Delta x)] \leq 1$$
$$-2 \leq -2\,(1-C)C\,[1-\cos(k\,\Delta x)] \leq 0$$
$$1 \geq (1-C)C\,[1-\cos(k\,\Delta x)] \geq 0$$

thus

$$\begin{aligned} 1 - C &\geq 0 \\ C &\leq 1 \\ \frac{v\,\Delta t}{\Delta x} &\leq 1 \end{aligned}$$

Thus for stablility we need to pick a time-step

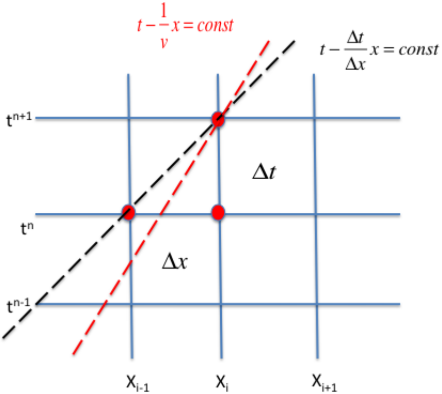
$$\Delta t \leq \frac{\Delta x}{v}$$

The stablility condition

$$\Delta t \leq \frac{\Delta x}{v}$$

implies that the numerical characteristics are contained within the physical characteristics since

$$\frac{\Delta t}{\Delta x} \leq \frac{1}{v}$$



# How does FTBS prevent the onset of instabilities?

Recall

$$\phi_i^{n+1} = \phi_i^n - C\,\left(\phi_i^n - \phi_{i-1}^n\right)$$

where $C \equiv \Delta t\, v/\Delta x$. Substitute

$$\begin{aligned}
\phi_i^{n+1} &= \phi_i^n + \Delta t\,\partial_t\phi_i^n + \frac{\Delta t^2}{2}\,\partial_t^2\phi_i^n + O(\Delta t^3) \\
\phi_{i-1}^n &= \phi_i^n - \Delta x\,\partial_x\phi_i^n + \frac{\Delta x^2}{2}\,\partial_x^2\phi_i^n + O(\Delta x^3)
\end{aligned}$$

then

$$\begin{aligned}
\phi_i^n + \Delta t\,\partial_t\phi_i^n + \frac{\Delta t^2}{2}\,\partial_t^2\phi_i^n &= \phi_i^n \\
-C\left[\phi_i^n - \phi_i^n + \Delta x\,\partial_x\phi_i^n - \frac{\Delta x^2}{2}\,\partial_x^2\phi_i^n\right]
\end{aligned}$$

Then

$$\Delta t\,\partial_t\phi + \frac{\Delta t^2}{2}\,\partial_t^2\phi = -v\frac{\Delta t}{\Delta x}\,\left[\Delta x\,\partial_x\phi - \frac{\Delta x^2}{2}\,\partial_x^2\phi\right]$$

$$\partial_t\phi + \frac{\Delta t}{2}\,\partial_t^2\phi = -v\left[\partial_x\phi - \frac{\Delta x}{2}\,\partial_x^2\phi\right]$$

$$\partial_t\phi + v\,\partial_x\phi + \frac{\Delta t}{2}\,\partial_t^2\phi - v\frac{\Delta x}{2}\,\partial_x^2\phi = 0$$

but from $\partial_t\phi = -v\,\partial_x\phi$ we have that

$$\partial_t^2\phi = -v\,\partial_t\partial_x\phi = -v\,\partial_x\partial_t\phi = v^2\,\partial_x^2\phi$$

thus

$$\partial_t\phi + v\,\partial_x\phi + v^2\frac{\Delta t}{2}\,\partial_x^2\phi - v\frac{\Delta x}{2}\,\partial_x^2\phi = 0$$

$$\partial_t\phi + v\,\partial_x\phi + \left(v^2\frac{\Delta t}{2} - v\frac{\Delta x}{2}\right)\,\partial_x^2\phi = 0$$

$$\partial_t \phi + v\,\partial_x \phi + \left( v^2 \frac{\Delta t}{2} - v \frac{\Delta x}{2} \right) \partial_x^2 \phi = 0$$

$$\partial_t \phi + v\,\partial_x \phi - v \frac{\Delta x}{2} \left( 1 - v \frac{\Delta t}{\Delta x} \right) \partial_x^2 \phi = 0$$

$$\partial_t \phi + v\,\partial_x \phi - v \frac{\Delta x}{2} (1 - C)\, \partial_x^2 \phi = 0$$

This equation has the form

$$\partial_t \phi + v\,\partial_x \phi - \alpha\, \partial_x^2 \phi = 0$$

advection-diffusion equation with

$$\alpha \equiv v \frac{\Delta x}{2} (1 - C)$$

Recall that for stability $C \leq 1$, thus $\alpha \geq 0$.

Given

$$\phi(t,x) = \phi_0\, e^{-pt}\, e^{-i\,k(x-q\,t)}$$

Substitution into

$$\partial_t\phi + v\,\partial_x\phi = 0 \quad\Rightarrow\quad p = 0 \quad q = v$$

$$\partial_t\phi + v\,\partial_x\phi - \alpha\,\partial_x^2\phi = 0 \quad\Rightarrow\quad p = \alpha\,k^2 \quad q = v \quad \text{dissipation}$$

$$\partial_t\phi + v\,\partial_x\phi - \beta\,\partial_x^3\phi = 0 \quad\Rightarrow\quad p = 0 \quad q = v - \beta\,k^2 \quad \text{dispersion}$$

- That is, the FTBS discretization introduces artificial numerical dissipation to prevent the growth of instabilities.
- Notice that the dissipation coefficient $\alpha \propto \Delta t$.
- Therefore, in the continuum limit $\lim_{\Delta x\to 0}\alpha = 0$

# Method of Lines

- The method of lines (MOL) is a numerical technique for solving PDEs by discretizing all the spatial derivatives.
- The net effect is translating the problem into an initial-value-problem with only one independent variable, time.
- The resulting system of ODEs (semi-discrete problem) is solved using sophisticated general purpose methods and software that have been developed for numerically integrating ODEs.

As an example, let's consider the advection equation

$$\partial_t \phi = -v\, \partial_x \phi$$

with $t \leq 0$, $v > 0$, and $x \in [0, 1]$. The initial data is $\phi(t = 0, x) = f(x)$ and the boundary condition $\phi(t, x = 0) = g(t)$.

- We first discretize the spatial derivative $\partial_x \phi$

$$\partial_x \phi|_i = \frac{\phi_{i+1} - \phi_{i-1}}{2\,\Delta x}$$

- Thus, the semi-discrete problem is

$$\frac{d\phi_i}{dt} = -v \frac{(\phi_{i+1} - \phi_{i-1})}{2\,\Delta x}$$

- Notice that we now have a coupled system of ODEs of the form

$$\frac{d\phi_i}{dt} = \rho(t, \phi_j)$$

  for which we can apply the methods we discussed before, in particular Runge-Kutta methods.

- Given that we are using center-space discretization, applying an Euler step (i.e. forward-time) will be unstable.

# Burger's Equation

- Recall the advection equation $\partial_t \phi + u\,\partial_x \phi = 0$ in which the quantity $\phi$ is advected or convected with a velocity $u$.
- Consider instead $\partial_t u + u\,\partial_x u = 0$. That is, the velocity at which the quantity is advected depends on the quantity itself.
- This equation is called the inviscid Burger's equation.
- This equation is widely used as a model to investigate non-linearities in fluid dynamics traffic control, etc..
- The general form of the Burger's equation is

$$\partial_t u + u\,\partial_x u = \nu \partial^2 u$$

with $\nu$ a viscosity coefficient.

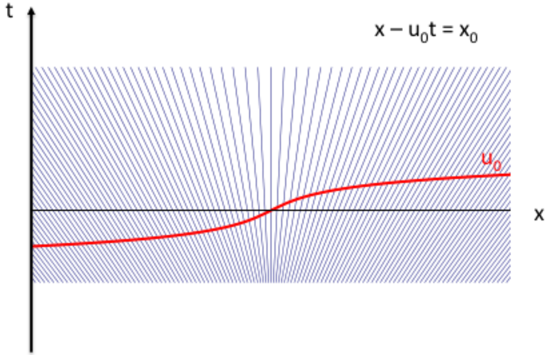
- Consider the inviscid Burger's equation $\partial_t u + u\,\partial_x u = 0$ with initial data $u(t=0,x) = u_0(x)$
- Method of Characteristics: Find the curves $x(t)$ tangent to the vector $\partial_t + u\,\partial_x$, such that $u(t, x(t))$ is constant.
- That is,

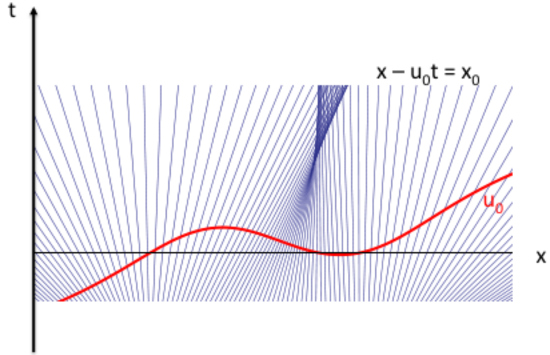
$$\begin{aligned} \frac{dx(t)}{dt} &= u(t, x(t)) \\ \frac{du(t, x(t))}{dt} &= \frac{\partial u}{\partial t} + \frac{dx}{dt}\frac{\partial u}{\partial x} \\ &= \frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial x} = 0 \end{aligned}$$

- The solutions are

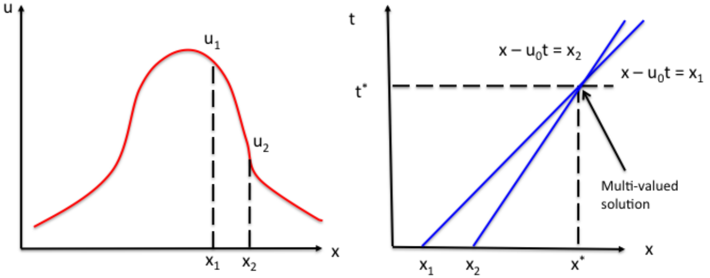
$$\begin{aligned} u(t, x(t)) &= u(0, x(0)) = u_0(x_0) \\ x(t) &= x_0 + t\,u(0, x(0)) = x_0 + t\,u_0(x_0) \end{aligned}$$

- Therefore, the solution to the Burger's equation reads $u(t,x) = u_0(x - t\,u_0(x_0))$
- Thus, the solution is constant along the characteristics $x_0 = x - t\,u_0(x_0)$.
- The characteristics are straight lines with slope $1/u_0(x_0)$ in the $t - x$ plain.
- For each characteristic, the value of the slope is fixed by the initial data $u_0(x)$ at $x = x_0$

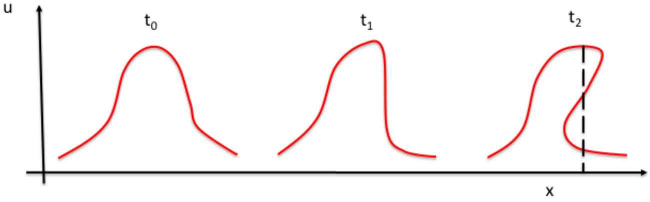

t
x – u0t = x0
u0
x


t
x − u_0t = x_0
u_0
x

Consider initial data of the form



The pulse evolves as




Notice that the larger the value of $u$ the more advected that portion of the solutions gets.

Let $S \equiv \partial_x u$, then

$$\begin{aligned} \frac{dS}{dt} &= \partial_t S + \frac{dx}{dt}\partial_x S = \partial_t S + u\,\partial_x S \\ &= \partial_t \partial_x u + u\,\partial_x^2 u = \partial_x(\partial_t u + u\,\partial_x u) - (\partial_x u)^2 \\ &= -S^2 \end{aligned}$$

The solution to this equation is

$$S = \frac{S_0}{t\,S_0 + 1} \quad \text{or} \quad \partial_x u = \frac{\partial_x u_0}{t\,\partial_x u_0 + 1}$$

Therefore, as $t \to -1/\partial_x u_0$ the slope of the solution diverges, that is, $\partial_x u \to \infty$. In other words, the solution develops a shock discontinuity.

In the case of the general viscous Burger's equation ($\nu \neq 0$), the shock profile gets smoothed out due to the dissipation.

# Shock Boundary

- Consider initial data such that $\partial_x^2 u_0(x) = 0$ everywhere and $\partial_x u_0(x) = \text{const} < 0$ if $x \in [x_1, x_2]$.
- Recall that the characteristics are given by the straight lines $x = \bar{x} + u_0(\bar{x})\, t$ where $\bar{x}$ is the value of $x$ at $t = 0$.
- Recall also that the shock will develop when $t^* = -1/\partial_x u_0(\bar{x})$.
- Therefore, the location where the shock develops is $x = \bar{x} + u_0(\bar{x})\, t^*$
- Consider to points $x_a, x_b$ such that $x_1 \le x_a, x_b \le x_2$
- Then

$$
\begin{aligned}
x_a + u_0(x_a)\, t^* &= x_b + (x_a - x_b) + [u_0(x_b) + (x_a - x_b)\partial_x u_0(x_b)]\ t^* \\
&= x_b + (x_a - x_b) + u_0(x_b)\, t^* - (x_a - x_b)\frac{\partial_x u_0(x_b)}{\partial_x u_0(x_b)} \\
&= x_b + u_0(x_b)\, t^*
\end{aligned}
$$

- Therefore, all the characteristics starting within the interval $[x_1, x_2]$ cross at the same point given by $x = \bar{x} - u_0(\bar{x})/\partial_x u_0(\bar{x})$

- Location of the shock boundary point $x = \bar{x} - u_0(\bar{x})/\partial_x u_0(\bar{x})$
- Notice that the characteristics have different slopes but the same shock developing time.
- Thus, the shape of the boundary shock depends on the “shape” of the initial data.