Computational Physics and Astrophysics

Partial Differential Equations:
Diffusive Initial Value Problems

Kostas Kokkotas
University of Tübingen, Germany

and

Pablo Laguna
Georgia Institute of Technology, USA

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Parabolic PDEs: Heat Equation

- Consider the 1D heat equation

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2}$$

with $D > 0$ in the domain $0 \leq x \leq L$ and $t \geq 0$.

- Boundary conditions:

$$\begin{aligned} u(0,t) &= c_1 \\ u(L,t) &= c_2 \end{aligned}$$

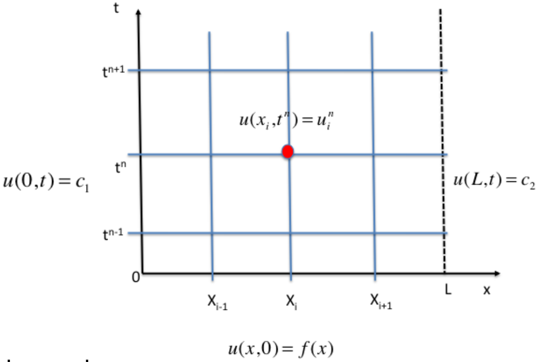
for $t \geq 0$

- Initial conditions :

$$u(x,0) = f(x)$$

for $0 \leq x \leq L$.

Consider the following computational domain:



Mesh spacing:

$$
\begin{aligned}
\Delta x &= x_{i+1} - x_i = L/(M-1) \quad i = 1, \cdots, M-1 \\
\Delta t &= t^{n+1} - t^n = ???
\end{aligned}
$$

Consider the following approximations

$$
\begin{aligned}
\partial_t u_i^n &= \frac{u_i^{n+1} - u_i^n}{\Delta t} - \frac{\Delta t}{2} \partial_t^2 u_i^n \\
\partial_x^2 u_i^n &= \frac{u_{i+1}^n - 2\, u_i^n + u_{i-1}^n}{\Delta x^2} - \frac{\Delta x^2}{12} \partial_x^4 u_i^n
\end{aligned}
$$

Therefore, the heat equation

$$\partial_t u_i^n - D\, \partial_x^2 u_i^n = 0$$

is approximated as

$$\frac{w_i^{n+1} - w_i^n}{\Delta t} - D\, \frac{w_{i+1}^n - 2\, w_i^n + w_{i-1}^n}{\Delta x^2} = 0$$

Notice that u_i^t is the solution to the exact equations, and w_i^n is the solution to the corresponding finite-difference approximation.

Truncation and Solution Errors

- Solution Error:

$$\epsilon_i^n = w_i^n - u_i^n$$

- Truncation Error:

$$\begin{aligned}
\tau_i^n &= \frac{u_i^{n+1} - u_i^n}{\Delta t} - D\,\frac{u_{i+1}^n - 2\,u_i^n + u_{i-1}^n}{\Delta x^2} \\
&= \partial_t u_i^n + \frac{\Delta t}{2}\partial_t^2 u_i^n - D\left(\partial_x^2 u_i^n + \frac{\Delta x^2}{12}\partial_x^4 u_i^n\right) \\
&= \partial_t u_i^n - D\,\partial_x^2 u_i^n + \frac{\Delta t}{2}\partial_t^2 u_i^n - D\,\frac{\Delta x^2}{12}\partial_x^4 u_i^n \\
&= \frac{\Delta t}{2}\partial_t^2 u_i^n - D\,\frac{\Delta x^2}{12}\partial_x^4 u_i^n
\end{aligned}$$

Solving for w_i^{n+1} in

$$\frac{w_i^{n+1} - w_i^n}{\Delta t} - D\,\frac{w_{i+1}^n - 2\,w_i^n + w_{i-1}^n}{\Delta x^2} = 0$$

one gets

$$w_i^{n+1} = \left(1 - \frac{2\,D\,\Delta t}{\Delta x^2}\right) w_i^n + \frac{D\,\Delta t}{\Delta x^2}\left(w_{i+1}^n + w_{i-1}^n\right)$$

or

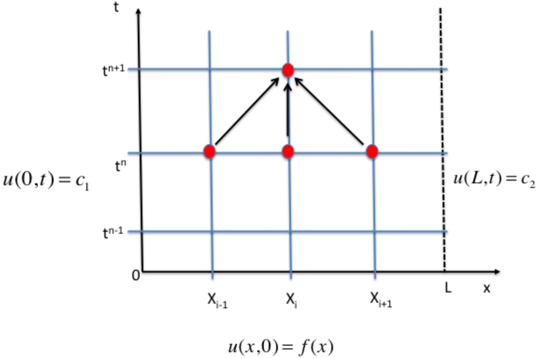
$$w_i^{n+1} = (1 - 2\,\alpha)\,w_i^n + \alpha\left(w_{i+1}^n + w_{i-1}^n\right)$$

where

$$\alpha \equiv \frac{D\,\Delta t}{\Delta x^2}$$

Forward-time, Center-Space Updating

$$w_i^{n+1} = (1 - 2\,\alpha)\, w_i^n + \alpha \left(w_{i+1}^n + w_{i-1}^n \right)$$



In matrix notation this is equivalent to $\mathbf{w}^{(n+1)} = \mathbf{A} \cdot \mathbf{w}^{(n)}$, where

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & & & & & \\ \alpha & (1-2\,\alpha) & \alpha & & & & \\ & \alpha & (1-2\,\alpha) & \alpha & & & \\ & & & \vdots & & & \\ & & \alpha & (1-2\,\alpha) & \alpha & & \\ & & & \alpha & (1-2\,\alpha) & \alpha \\ & & & & 0 & 1 \end{pmatrix}$$

$$\mathbf{w}^{(n)} = \begin{pmatrix} w_1^n \\ w_2^n \\ w_3^n \\ \vdots \\ w_{M-2}^n \\ w_{M-1}^n \\ w_M^n \end{pmatrix} \qquad \mathbf{w}^{(0)} = \begin{pmatrix} c_1 \\ f_2 \\ f_3 \\ \vdots \\ f_{M-2} \\ f_{M-1} \\ c_2 \end{pmatrix}$$

Stability

Consider a small error in the initial data. Let's investigate how this error propagates.

$$\begin{aligned}
\mathbf{w}^{(1)} &= \mathbf{A}\cdot\left(\mathbf{w}^{(0)}+\mathbf{e}^{(0)}\right) = \mathbf{A}\cdot\mathbf{w}^{(0)} + \mathbf{A}\cdot\mathbf{e}^{(0)} \\
\mathbf{w}^{(2)} &= \mathbf{A}\cdot\mathbf{w}^{(1)} = \mathbf{A}^2\cdot\mathbf{w}^{(0)} + \mathbf{A}^2\cdot\mathbf{e}^{(0)} \\
&\vdots \\
\mathbf{w}^{(n)} &= \mathbf{A}\cdot\mathbf{w}^{(n-1)} = \mathbf{A}^n\cdot\mathbf{w}^{(0)} + \mathbf{A}^n\cdot\mathbf{e}^{(0)}
\end{aligned}$$

A method is said to be stable if

$$||\mathbf{A}^n\cdot\mathbf{e}^{(0)}|| \leq ||\mathbf{e}^{(0)}||$$

Thus, we must require that

$$||\mathbf{A}^n|| \leq 1$$

Von Neumann Stability Analysis of FTCS

Substitute $w_i^n = \xi^n e^{I\,k\,i\,\Delta x}$ into

$$w_i^{n+1} = (1 - 2\,\alpha)\,w_i^n + \alpha\left(w_{i+1}^n + w_{i-1}^n\right)$$

So,

$$\begin{aligned}
\xi^{n+1}\,e^{I\,k\,i\,\Delta x} &= (1-2\,\alpha)\,\xi^n\,e^{I\,k\,i\,\Delta x} + \alpha\left(\xi^n\,e^{I\,k\,(i+1)\,\Delta x} + \xi^n\,e^{I\,k\,(i-1)\,\Delta x}\right) \\
\xi &= (1-2\,\alpha) + \alpha\left(e^{I\,k\,\Delta x} + e^{-I\,k\,\Delta x}\right) \\
\xi &= 1 - 2\,\alpha + 2\,\alpha\cos(k\,\Delta x) = 1 - 2\,\alpha\,[1 - \cos(k\,\Delta x)] \\
\xi &= 1 - 4\,\alpha\,\sin^2(k\,\Delta x/2)
\end{aligned}$$

Then $|\xi| \leq 1$ implies

$$\begin{aligned}
-1 &\leq 1 - 4\,\alpha\,\sin^2(k\,\Delta x/2) \leq 1 \\
-2 &\leq -4\,\alpha\,\sin^2(k\,\Delta x/2) \leq 0 \\
1/2 &\geq \alpha \geq \alpha\,\sin^2(k\,\Delta x/2) \geq 0
\end{aligned}$$

Von Neumann Stability Analysis of FTCS

That is, in order to be stable one requires that *alpha* $\leq 1/2$. But since $\alpha \equiv D\,\Delta t/\Delta x^2$. This condition implies that

$$\Delta t \leq \frac{\Delta x^2}{2\,D}$$

Notice that $\Delta t \propto \Delta x^2$; thus, halving the grid-spacing will decrease the time-step by a factor of 1/4.
Also if

$$u(x,t) = u_0\, e^{-pt}\, e^{-i\,k(x-q\,t)}$$

then

$$\partial_t u - D\,\partial_x^2 u = 0$$

yields $p = D\,k^2$ and $q = 0$. Thus,

$$u(x,t) = u_0\, e^{-D\,k^2\,t}\, e^{-i\,k\,x}$$

Notice that diffusion is stronger for large D or small scale features, i.e. large wave-numbers k.

In other words, the diffusion time τ across a spatial scale λ is

$$\partial_t u - D\,\partial_x^2 u = 0 \quad \Rightarrow \quad \frac{u}{\tau} - D\,\frac{u}{\lambda^2} \sim 0$$

thus,

$$\tau \sim \frac{\lambda^2}{D}$$

Therefore, given the condition $\Delta t \leq \Delta x^2/D$, evolving scales $\lambda \gg \Delta x$ will require to take number of steps of order

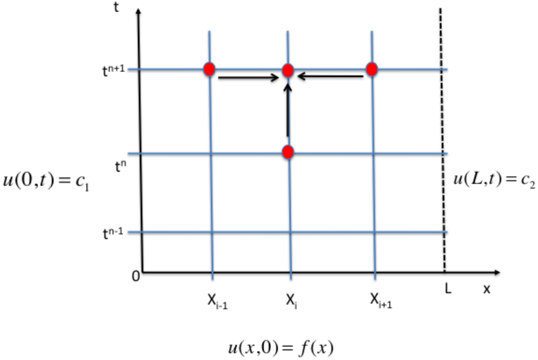
$$\frac{\tau}{\Delta t} \sim \frac{\lambda^2}{D\,\Delta t} \sim \frac{\lambda^2}{\Delta x^2} \gg 1$$

which could be computationally prohibited.

Fully Implicit Evolution

Consider the following discretization

$$\frac{w_i^{n+1} - w_i^n}{\Delta t} - D\,\frac{w_{i+1}^{n+1} - 2\,w_i^{n+1} + w_{i-1}^{n+1}}{\Delta x^2} = 0$$



Solving for w_i^n, we get

$$(1 - 2\,\alpha)\, w_i^{n+1} - \alpha \left(w_{i+1}^{n+1} + w_{i-1}^{n+1} \right) = w_i^n$$

where again $\alpha \equiv \frac{D\,\Delta t}{\Delta x^2}$.
A von Neumann stability analysis yields

$$\xi = \left[1 + 4\,\alpha\, \sin^2 (k\,\Delta x/2) \right]^{-1}$$

That is, $|\xi| < 1$ for any step size Δt. The scheme is unconditionally stable

In matrix notation, the implicit update is equivalent to $\mathbf{A} \cdot \mathbf{w}^{(n+1)} = \mathbf{w}^{(n)}$, where

$$
\mathbf{A} = \begin{pmatrix}
1 & 0 & & & & & \\
-\alpha & (1-2\,\alpha) & -\alpha & & & & \\
 & -\alpha & (1-2\,\alpha) & -\alpha & & & \\
 & & & \vdots & & & \\
 & & -\alpha & (1-2\,\alpha) & -\alpha & & \\
 & & & -\alpha & (1-2\,\alpha) & -\alpha & \\
 & & & & 0 & 1 &
\end{pmatrix}
$$

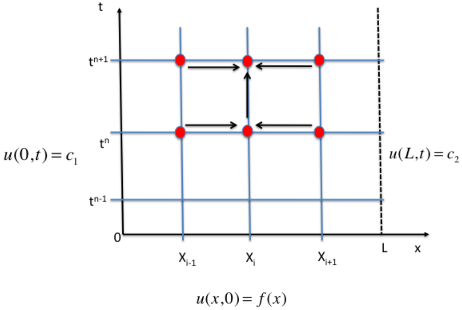
$$
\mathbf{w}^{(n)} = \begin{pmatrix} w_1^n \\ w_2^n \\ w_3^n \\ \vdots \\ w_{M-2}^n \\ w_{M-1}^n \\ w_M^n \end{pmatrix}
\qquad
\mathbf{w}^{(0)} = \begin{pmatrix} c_1 \\ f_2 \\ f_3 \\ \vdots \\ f_{M-2} \\ f_{M-1} \\ c_2 \end{pmatrix}
$$

- Therefore, at every step, instead of updating the solution with $\mathbf{w}^{(n+1)} = \mathbf{A} \cdot \mathbf{w}^{(n)}$ (explicit update), now one needs to solve the system $\mathbf{A} \cdot \mathbf{w}^{(n+1)} = \mathbf{w}^{(n)}$ (implicit update)
- In spite of been able to take large steps, the implicit update could be quite expensive.
- In addition, the fully implicit method is still only first order accurate in time.
- Can we design a scheme that is second order accurate in both space and time with the stability properties of the implicit method?

Crank-Nicholson

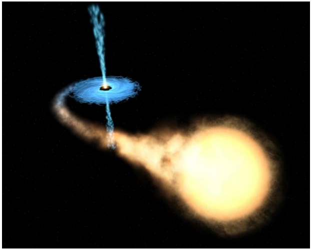
$$\frac{w_i^{n+1} - w_i^n}{\Delta t} = \frac{D}{2}\left[\frac{w_{i+1}^{n+1} - 2\,w_i^{n+1} + w_{i-1}^{n+1}}{\Delta x^2} + \frac{w_{i+1}^n - 2\,w_i^n + w_{i-1}^n}{\Delta x^2}\right]$$

$$2\,(1+\alpha)\,w_i^{n+1} - \alpha\left(w_{i+1}^{n+1} + w_{i-1}^{n+1}\right) = -2\,(1-\alpha)\,w_i^n + \alpha\left(w_{i+1}^n + w_{i-1}^n\right)$$



Thin Accretion Disks

- An accretion disk formes when by diffuse material orbits a central body.
- Gravity and losses in angular momentum causes the material ito spiral inward towards the central body.
- Gravitational forces compress the material causing the emission of electromagnetic radiation.
- Accretion disks of young stars and protostars radiate in the infrared; those around neutron stars and black holes in the x-ray.



- Accration condition:

$$\begin{aligned} \text{central object} &\quad \Leftarrow \quad \text{mass flow} \\ \text{central object} &\quad \Rightarrow \quad \text{angular momentum flow} \end{aligned}$$

- Angular momentum per unit mass:

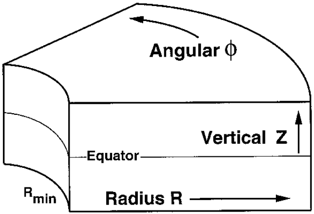
$$j = r^2\,\omega$$

- Kepler's 3th law:

$$\omega = \sqrt{\frac{G\,M}{r^3}}$$

- Therefore,

$$j = \sqrt{G\,M\,r}$$



- We will use cylindrical coordinates $\{t, r, \phi, z\}$.
- We will not include the central object in the computation. That is $r_{min} \leq r \leq r_{max}$

Basic Assumptions:

- The viscosity in the disk drives the flow of angular momentum outwards.
- The size of the disk is much larger than its thickness, i.e. thin disk
- All physical properties are independent of ϕ

Disk Model

- z-dependences will be integrated over the disk thickness. For example,

$$\sigma = \int_{-\infty}^{\infty} \rho \, dz \qquad \text{surface mass density}$$

- Keplerian disk

$$v_\phi = r\,\omega \qquad \text{where} \qquad \omega = \sqrt{\frac{G\,M}{r^3}}$$

- Mass of the disk is much smaller than the mass of the central object.

$$2\,\pi \int_0^{\infty} \sigma\, r\, dr \ll M$$

- The orbital speed of a fluid element in the disk is much larger than the thermal speed.

$$v_\phi = r\,\omega \gg \sqrt{\frac{kt}{m}}$$

Disk Properties

- Mass Density: ρ or σ
- Fluid Velocity: $\vec{v} = (v_r, v_\phi, v_z) = (v_r, r\,\omega, 0)$
- Energy Density: ignored
- Gas pressure: ignored
- Disk viscosity: ν
- Disk self-gravity: ignored
- Central object gravity: M/r

Mass Conservation

Also known as the continuity equation

$$\partial_t \rho + \nabla \cdot (\rho \, \vec{v}) = 0$$

In cylindrical coordinates, it reads

$$\begin{aligned}
&\partial_t \rho + \frac{1}{r}\partial_r(\rho \, r \, v_r) + \partial_\phi(\rho \, v_\phi) + \partial_z(\rho \, v_z) = 0 \\
&\partial_t \rho + \frac{1}{r}\partial_r(\rho \, r \, v_r) = 0 \\
&\int_{-\infty}^{\infty} \left[\partial_t \rho + \frac{1}{r}\partial_r(\rho \, r \, v_r)\right] dz = 0
\end{aligned}$$

Then

$$\partial_t \sigma + \frac{1}{r}\partial_r(\sigma \, r \, v_r) = 0$$

Angular Momentum Conservation

$$\partial_t(\rho j) + \frac{1}{r}\partial_r(r\,\rho\,j\,v_r) = -\frac{1}{r}\partial_r(r\,\Pi_{r\rho})$$

$$\text{where} \qquad \Pi_{r\phi} = -\nu\,\rho\,r^2\frac{d\omega}{dr}$$

Integration across the z-direction and substitution mass conservation equation yields

$$\partial_t j + v_r\partial_r j = \frac{1}{r\,\sigma}\partial_r\left(r^3\,\nu\,\sigma\,\frac{d\omega}{dr}\right)$$

Recall, $j = \sqrt{G\,M\,r}$ and $\omega = \sqrt{G\,M/r^3}$.

$$v_r = \frac{1}{r\,\sigma}\left(\frac{d\,j}{dr}\right)^{-1}\partial_r\left(r^3\,\nu\,\sigma\,\frac{d\omega}{dr}\right)$$

$$v_r = \frac{-3}{r^{1/2}\,\sigma}\partial_r\left(r^{1/2}\,\nu\,\sigma\right)$$

We then have

$$\partial_t\sigma + \frac{1}{r}\partial_r(\sigma\, r\, v_r) = 0$$
$$v_r = \frac{-3}{r^{1/2}\,\sigma}\partial_r\left(r^{1/2}\,\nu\,\sigma\right)$$

to solve for σ and v_r given a viscosity model for $\nu = \nu(\sigma, r, t)$. Notice that the second equation trivially gives a way to compute the drift velocity.

Substitution of v_r in the mass conservation equation yields

$$\partial_t\sigma - \frac{3}{r}\partial_r\left[r^{1/2}\partial_r\left(r^{1/2}\,\nu\,\sigma\right)\right] = 0$$

What type of equation is

$$\partial_t \sigma - \frac{3}{r}\partial_r \left[r^{1/2} \partial_r \left(r^{1/2}\, \nu\, \sigma \right) \right] = 0$$

Define $x \equiv 2\, r^{1/2}$ and $\mu \equiv \nu\, \sigma\, r^{1/2}$. One gets,

$$\partial_t \mu - \frac{12\, \nu}{x^2} \partial_x^2 \mu = 0$$

This is a diffusion equation with diffusion coefficient

$$D = \frac{12\, \nu}{x^2}$$

On the tidal interaction between protoplanets and the protoplanetary disk, D.N.C. Lin and John Paapaloizou, Astrophysical Journal, vol 309, p 846, y 1986.

- Viscosity model for solar disks $\nu = \sigma^2$
- Tidal interaction

$$v_r = \frac{-3}{r^{1/2}\,\sigma}\partial_r\left(r^{1/2}\,\nu\,\sigma\right) + 3\,\Lambda\,r^{1/2}$$

where Λ is the injection rate of angular momentum by tidal interaction with the protoplanet.

$$\Lambda = \mathrm{sign}(r - r_p)\frac{A}{r}\left(\frac{r}{\delta_p}\right)^4$$

where r_p is the radial position of the protoplanet, A is the strength of tidal effects and $\delta_p = Max(H, |r - r_p|)$

Therefore

$$\partial_t \sigma = -\frac{1}{r}\partial_r(\sigma\, r\, v_r) \quad \text{and} \quad v_r = \frac{-3}{r^{1/2}\,\sigma}\partial_r\left(r^{1/2}\,\sigma^3\right) + 3\,\Lambda\, r^{1/2}$$

yields

$$\partial_t \sigma = \frac{3}{r}\partial_r\left[r^{1/2}\partial_r\left(r^{1/2}\sigma^3\right) - \Lambda\,\sigma\, r^{3/2}\right]$$

or

$$\partial_t \sigma = \frac{3}{r}\partial_r\left[r^{1/2}\partial_r\left(r^{1/2}\sigma^3\right) - \mathrm{sign}(r - r_p) A\,\sigma\, r^{1/2}\left(\frac{r}{\delta_p}\right)^4\right]$$

We need the equation of motion for the protoplanet. In the absence of the disk, we will assume that the planet moves in circular orbits. That is, $\frac{dr_p}{dt} = 0$.

The presence of the disk modifies the protoplanet orbital dynamics as follows:

$$\frac{d\, r_p}{dt} = -3\, B\, A\, r_p^{1/2} \int_{r_{min}}^{r_{max}} \mathrm{sign}(r - r_p)\, \sigma \left(\frac{r}{\delta_p}\right)^4 dr$$

where B is the mass ratio of the disk to the protoplanet.

We will consider the following parameters

$$\begin{aligned} A &= 10^{-3} \\ B &= 20 \\ r_{min} &= 0 \\ r_{max} &= 2 \\ r_p &= 0.8 \\ \sigma &= e^{-r^2/2} \quad \text{at} \quad t = 0 \\ H &= 0.05 \end{aligned}$$

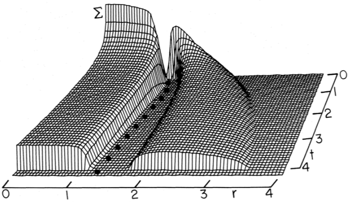


FIG. 4a

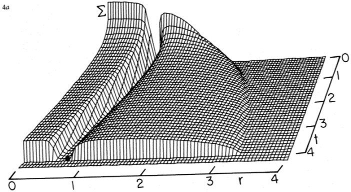


FIG. 4b

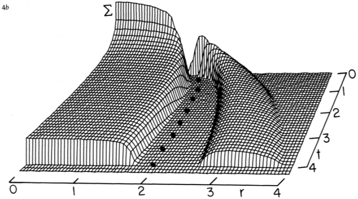


FIG. 4.—Evolution of the disk's Σ distribution and the protoplanet's orbit. For every case, $A = 10^{-3}$, $B = 20$, and $h_0 = 0.05$. Initial $r_p =$ (g) 0.6, (b) 0.7, and (c) 0.8. At $\tau = 0$, $\Sigma = 1$ for $r \leq 1$ and 0 elsewhere. The location of the protoplanet is indicated by a filled circle.