So there are two ODEs:
theta_dot(t) = omega(t) = F
omega_dot(t) = -sin(theta(t)) = G

Now when they apply RK2 method for F then its straightforward with the expansion terms,

k1_theta = omega_n(t)
k2_theta = omega_n(t) + (h/2)*k1_theta

But for G it should be,

k1_omega = G(theta_n(t)) = -sin(theta_n(t))
k2_omega = G(theta_n(t) + (h/2)*k1_omega) = -sin(theta_n(t) + (h/2)*k1_omega)

Instead, from what I understand is, these students wrote:
k2_omega = -sin(theta_n(t)) - sin((h/2)*k1_omega)

REVIEW

ODE's $\quad \frac{dy_i(x)}{dx} = f_i(x, y_1, \ldots, y_N)$ $\quad$ set of N 1st order equations

any ODE can be rewritten in 1st order

$$\frac{d^2y}{dx^2} + q(x)\frac{dy}{dx} = r(x)$$

$f(t+\Delta t) = f(t) + hf'(t) + \frac{h^2 f''(t)}{2!}$

why?

Example $\quad \frac{dy_i}{dx} = f_i$

$$= \frac{y_{i+1} - y_i}{\Delta x} + O(\Delta x^2)$$

$$y_{i+1} = y_i + \Delta x f_i + \underbrace{O(\Delta x^2)}_{\text{this is the local truncation error}}$$

as previously computed

$O(\Delta x)$ after N steps

$$N \propto \frac{x_{max} - x_0}{\Delta x}$$

why?

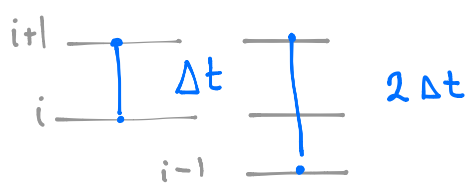
How do you make it more accurate?

$f(t+\Delta t) = f(t) + hf' + h^2 f''/2$

$f(t-\Delta t) = f(t) - hf' + h^2 f''/2$

center difference $\quad \frac{dy}{dx} = \frac{y_{i+1} - y_{i-1}}{2\Delta x} + O(x^3)$

over N iterations $O(\Delta x^2)$



# Convergence Test

$$e(t^n) = y(t^n) - y_{exact}(t^n)$$

$$y(t^n) - y_{exact}(t^n) \underset{\Delta t \to 0}{\sim} \text{for Euler?} \sim \Delta t^2 \; \text{function}(t^n)$$

depends on method!

$$y(t^n) \sim y_{exact}(t^n) - \Delta t^2 \, \text{function}(t^n)$$

now we need ratios

coarse — $nt = 2^{\ell} + 1$
medium — $nt = 2^{\ell+1} + 1$
fine — $nt = 2^{\ell+2} + 1$

$$y_\ell(t^n) \sim y_{exact}(t^n) - (\Delta t_\ell)^2 \, \text{func}(t^n)$$

SAME!

$$y_{\ell+1}(t^n) \sim y_{exact}(t^n) - (\Delta t_{\ell+1})^2 \, \text{func}(t^n)$$

$$y_{\ell+2}(t^n) \sim y_{exact}(t^n) - (\Delta t_{\ell+2})^2 \, \text{func}(t^n)$$

$$\Delta t_{\ell+1} = \frac{\Delta t_\ell}{2} \quad ; \quad \Delta t_{\ell+2} = \frac{\Delta t_\ell}{4}$$

TRUE INDEPENDENT OF METHOD

$$\Delta t = \frac{t_{max}}{nt - 1} = 2^{-\ell} \, t_{max}$$

hold $t_{max}$ constant

$$\Delta t_{\ell+1} = 2^{(-\ell-1)} \, t_{max} = \frac{2^{-\ell}}{2} \, t_{max} \quad \therefore \quad \Delta t_{\ell+1} = \frac{\Delta t_\ell}{2}$$

What is $y_\ell(t^n) - y_{\ell+1}(t^n)$ ?

$$y_{exact}(t^n) - (\Delta t_\ell)^2 \text{func}(t^n) - y_{exact}(t^n) + (\Delta t_{\ell+1})^2 \text{func}(t^n) = -(\Delta t_\ell^2 - \Delta t_{\ell+1}^2) f$$

$$= -\left(\Delta t_\ell^2 - \frac{\Delta t_\ell^2}{4}\right) \text{func}(t^n) = -\frac{3}{4} \Delta t_\ell^2 \, \text{func}(t^n)$$

$$y_{\ell+1} - y_{\ell+2} = y_{exact} - (\Delta t_{\ell+1})^2 \text{func} - y_{exact} + (\Delta t_{\ell+2})^2 \text{func}$$

$$= -\left(\frac{\Delta t_\ell^2}{4} \text{func} - \frac{\Delta t_\ell^2}{16} f\right) \quad -\frac{3}{16} \Delta t_\ell^2 \, \text{func}(t^n)$$

$$\frac{y_\ell - y_{\ell+1}}{y_{\ell+1} - y_{\ell+2}} = \frac{-\frac{3}{4}\Delta t_\ell^2 \text{func}(t^n)}{-\frac{3}{16}\Delta t_\ell^2 \text{func}(t^n)} = 4 \text{ as a function of time}$$

NOT Euler ② $O(h)$

Only for a method $O(\Delta t^2)$!

RK 4 is 4th order [global]

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$k_1 \propto h$
$k_2 \propto h^2$
$k_3 \propto h^3$
$k_4 \propto h^4$

$$e(t^n) = y(t^n) - y_{exact}(t^n)$$

$$y(t^n) - y_{exact}(t^n) \underset{\Delta t \to 0}{\sim} \Delta t^4 \text{ func}(t^n)$$

$$y(t^n) \sim y_{exact}(t^n) - \Delta t^4 \text{ function}(t^n)$$

$$y_{exact}(t^n) - (\Delta t_\ell)^4 \text{func}(t^n) - y_{exact}(t^n) + (\Delta t_{\ell+1})^4 \text{func}(t^n) = -(\Delta t_\ell^4 - \Delta t_{\ell+1}^4) f$$
$$= -\left(\Delta t_\ell^4 - \frac{\Delta t_\ell^4}{16}\right)\text{func}(t^n) = \frac{-15}{16}\Delta t_\ell^4 \text{ func}(t^n)$$

$$y_{\ell+1} - y_{\ell+2} = y_{exact} - (\Delta t_{\ell+1})^4 \text{func} - y_{exact} + (\Delta t_{\ell+2})^4 \text{func}$$
$$= -\left(\frac{\Delta t_\ell^4}{16}\text{func} - \frac{\Delta t_\ell^4}{256} f\right) = \frac{-15}{256}\Delta t_\ell^4 \text{ func}(t^n)$$

$$\frac{y_\ell - y_{\ell+1}}{y_{\ell+1} - y_{\ell+2}} = \frac{-\frac{15}{16}\Delta t_\ell^4 \text{func}(t^n)}{-15/256\,\Delta t_\ell^4 \text{func}(t^n)} = 16 \quad (+1$$

# ACCURACY $\neq$ STABILITY

Convergence: $h \to 0$ recover $y_{exact}$

Stability: are small perturbation amplified?

exp. $\frac{dy}{dx} = -cy$ ; $c>0$     soln? $y(x) = e^{-cx}$

$\frac{y_{n+1} - y_n}{h} = -cy_n \Rightarrow y_{n+1} = y_n - hcy_n = (1-hc)y_n$   SPECIFIC

STABLE?

Assume $y_n = A\xi^n$ amplification factor   $|\xi| \leq 1$ stable criteria

$$y_{n+1} = y_n(1-hc)$$
$$A\xi^{n+1} = A\xi^n(1-hc)$$
$$\xi = 1-hc$$

$$-1 < \xi < 1$$

here $-1 < 1-hc < 1$

$-2 < -hc < 0$

when $hc < 2$ Euler is stable

$h < 2/c \longrightarrow$ SPACE ACCORDINGLY!

Pendulum $\frac{d\theta}{dt} = w$ ; $\frac{dw}{dt} = -\sin\theta$

$$\theta^{n+1} = \theta^n + \Delta t\, w^n$$

$$\theta^n = A\xi^n$$

$$A\xi^{n+1} = A\xi^n + \Delta t\, w^n$$

$$A(\xi^{n+1} - \xi^n) = \Delta t\, w^n$$

$$|\xi| = |\Delta t\, w^n / A| \leq 1$$

$$w^{n+1} = w^n - \Delta t \sin\theta^n$$

$$A(\xi^{n+1} - \xi^n) = -\Delta t \sin\theta^n$$

$$A\xi = -\Delta t \sin\theta^n$$

$$|\xi| = \left|\frac{\Delta t \sin\theta^n}{A}\right| \leq 1$$

Small angle approximation $\quad \sin\theta \sim \theta$

$$\begin{pmatrix} \theta_{n+1} \\ w_{n+1} \end{pmatrix} = \begin{pmatrix} 1 & \Delta t \\ -\Delta t & 1 \end{pmatrix} \begin{pmatrix} \theta_n \\ w_n \end{pmatrix}$$

$\lambda$ eigenvalues of $2 \times 2$ matrix $\quad \lambda \leq 1$

$$\lambda^2 - 2\lambda + 1 + \Delta t^2 = 0$$

$$\lambda = \frac{-2 \pm \Delta t \sqrt{-4}}{2}$$

unstable

$$M = \begin{pmatrix} 1 & \Delta t \\ -\Delta t & 1 \end{pmatrix}$$

$\det(M - \lambda I)$

$$M - \lambda I = \begin{pmatrix} 1-\lambda & \Delta t \\ -\Delta t & 1-\lambda \end{pmatrix} ; \quad \det(M - \lambda I) = (1-\lambda)(1-\lambda) - (\Delta t)(-\Delta t)$$

$$= 1 - 2\lambda + \lambda^2 + \Delta t^2$$