Lecture Notes for PHYS 527 Fall 2007

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11 Lecture: Spectral Analysis

We take a quick return to data analysis using spectral methods.

11.1 Discrete Fourier Transform

- A Fourier transform is a mathematical formula that transform signals between time (or space) and frequency domain. It is reversible.

$$\tilde{F}(f) = \int_{-\infty}^{\infty} F(t) e^{-2\pi i t f} dt$$

$$Y(f) = \int_{-\infty}^{\infty} y(t) e^{-2\pi i t f} dt$$

or

$$F(t) = \int_{-\infty}^{\infty} \tilde{F}(f) e^{+2\pi i t f} df$$

$$y(t) = \int_{-\infty}^{\infty} Y(f) e^{+2\pi i t f} df$$

- For a periodic function over time, the Fourier transform is simplified to a calculation of discrete set of complex amplitudes (Fourier series coeffs).
- Periodic or oscillatory functions require different curve fitting than linear or polynomial, some kind of trigonometric fitting - spectral analysis
- Spectral analysis can easily be a semester course, usually called signal processing in Engineering.
- We shall only cover the basics, see Jenkins and Watts for complete study.
- Given a vector of N points, and data set $\mathbf{y} = [y_1, y_2, \ldots . y_N]$ that is typically a time series.

- The data is evenly spaced in time so $t_{j+1} = \tau j$ and τ is the sampling interval and $j = 0, ..N - 1$.
- The vector $\mathbf{Y}$ is the discrete Fourier transform of $\mathbf{y}$

$$Y_{k+1} = \sum_{j=0}^{N-1} y_{j+1} e^{-2\pi ijk/N}$$

  where $i = \sqrt{-1}$ and $k = 0 \ldots N - 1$.

- The inverse transform is

$$y_{k+1} = \frac{1}{N} \sum_{k=0}^{N-1} Y_{k+1} e^{2\pi ijk/N} \, .$$

- NOTE: different packages and libraries define this transformation differently, especially the normalization. Always check.
- Each point, Y_{k+1}, of the transform has an associated frequency

$$f_{k+1} = \frac{k}{\tau N} \, .$$

  The lowest, non-zero, frequency is at $k = 1$, $f_2 = 1/(\tau N) = 1/T$ where T is the length of the time series.

- This implies that to measure very low frequencies, we need very long time series.
- The highest frequency is $f_N \approx 1/\tau$, so to measure very high frequencies we need to use a very short sampling rate. (Aliasing in a second)

11.1.1 Example

- Initialize a sine wave time series, $y_{j+1} = sin(2\pi f_s j\tau + \phi_0)$
- Compute the transform using either Direct sum of fast Fourier transform
- Compute the power spectrum, $P_{k+1} = \mid Y_{k+1} \mid^2$
- Example 1: $\tau = 1$, $N = 50$, $f_s = 0.2$ and $\phi_0 = 0$. Use `ftdemo.m` to see this. The sine wave is jagged because of the slow sampling rate. The real part of the transform is 0 and the imaginary part has 2 spikes, one at f=0.2 and another at f=0.8 ($k = 10$ and $k = 40$) .
- Example 2: If we change the phase to $\phi_0 = \pi/2$, we get a cosine wave and the imaginary part is zero and the real part now has two spikes at f=0.2 and 0.8.

- Example 3: If we choose a frequency that does not fall on a grid point, $f_s \neq f_{k+1}$ such as $f_s = 0.2123$ the transform is not as simple. Now we have both real and imaginary parts in addition to a spread of the spikes. This is because the frequency of the signal is not equal to nor a multiple of $1/\tau N$.
- At this point, it is typical to plot the power spectrum instead

$$P_{k+1} = \mid Y_{k+1} \mid^2 = Y_{k+1} Y_{k+1}^*$$

  where $*$ is the complex conjugate.
- Now we see two well defined spikes, one of which is between 0.2 and 0.22 as expected.
- Why two spikes?

11.1.2 Aliasing and Nyquist Frequency

- If we do a fourth example, and set $f_s = 0.8$, and keep everything else fixed, we will get the same result as $f_s = 0.2$ despite the fact that the frequencies are completely different. The only difference is a phase shift of π
- The Fourier transforms for the two sine waves, one at $f_s = 0.2$ and the one of $f_s = 0.8$ for $\tau = 1$ are the same for that phase shift.
- This is known as aliasing.
- Aliasing causes a limit to how high of a frequency we can actual resolve given a sample time, τ. This upper bound is called the Nyquist frequency

$$f_{Ny} = \frac{1}{2\tau}\,.$$

- For the example we just did, $\tau = 1$ and the Nyquist frequency is 1/2.
- We basically are truncating our Fourier transform at this upper bound, and are discarding the upper half of the vector $\mathbf{Y}$.
- We can understand this through an "information" argument. The data, $\mathbf{y}$ is a real array containing N points. The Fourier transform is complex, containing N points real and N points imaginary. It contains a duplicate of the signal in the upper half of $\mathbf{Y}$.

11.2 Fast Fourier Transform

The FFT is an efficient computational tool. The physical process is that the same function can be represented in two ways

a) time domain h(t)

b) frequency domain H(f).

The means of going from h(t) to H(f) is the Fourier Transform

$$H(f) \equiv \int_{-\infty}^{\infty} h(t)e^{2\pi ift}dt \tag{1}$$

$$h(t) \equiv \int_{-\infty}^{\infty} H(f)e^{-2\pi ift}df \tag{2}$$

The convolution theorem states that convolution in one domain is just the point-wise multiplication in the other:

$$g * h \equiv \int_{-\infty}^{\infty} g(\tau)h(t-\tau)d\tau = G(f)H(f)$$

and likewise the correlation theorem

$$corr(g,h) \equiv \int_{-\infty}^{\infty} g(\tau+t)h(\tau)d\tau = G(f)H^*(f)$$

where $H^*(f)$ is the complex conjugate of $H(f)$.

Here are some handy facts

If...	then ...
h(t) is real	$H(-f) = H^*(f)$
h(t) is imaginary	$H(-f) = -H^*(f)$
h(t) is even	$H(-f) = H(f)$
h(t) is odd	$H(-f) = -H(f)$
h(t) is real and even	H(f) is real and even
h(t) is real and odd	H(f) is imaginary and odd
h(t) is imaginary and even	H(f) is imaginary and even
h(t) is imaginary and odd	H(f) is real and odd

- “time-scaling”

$$h(at) = \frac{1}{|a|}H(f/a)$$

- "frequency-scaling"

$$\frac{1}{|b|}h(t/b) = H(bf)$$

- "time-shifting"

$$h(t - t_0) = H(f)e^{2\pi i f t_0}$$

- "frequency-shifting"

$$h(t)e^{-2\pi i f_0 t} = H(f - f_0)$$

Wiener-Khinchin Theorem

$$Corr(g, g) = |G(f)|^2$$

this is auto-correlation

Parseval's theorem

$$\text{Total Power} \equiv \int_{-\infty}^{\infty} |h(t)|^2 dt = \int_{-\infty}^{\infty} |H(f)|^2 df$$

One-sided power spectral density

$$P_h(f) \equiv |H(f)|^2 + |H(-f)|^2 \text{ for } 0 \leq f < \infty$$

11.2.1 Fourier Transform of discretely sampled data

Nyquist frequency is

$$f_c \equiv \frac{1}{2\Delta}$$

Critical sampling $= \frac{2points}{cycle}$

SAMPLING THEOREM: If h(t), sampled at an interval, Δ, is such that $H(f) = 0$ for all $|f| \geq f_c \equiv 1/2\Delta$ then h(t) is completely determined by h_n

$$h(t) = \Delta \sum_{n=-\infty}^{+\infty} h)n \frac{sin[2\pi f_c(t - n\Delta)]}{\pi(t - n\Delta)}$$

If h(t) is not bandwidth limited, i.e. $H(f) \neq 0$ for all $|f| < f_c$, then ALIASING

11.3 Discrete Fourier Transform

$h_k \equiv h)(t_k)$ where $t_k \equiv k\Delta$, and $k = 0, 1, 2, \ldots, N-1$ and N is even.

Goal is to estimate H(f) at

$$f_n \equiv \frac{n}{N\Delta}$$

where $n = -\frac{N}{2}, \ldots, \frac{N}{2}$ so $-f_c \leq f_n \leq f_c$

$$H(f_n) = \int_{-\infty}^{\infty} h(t)e^{2\pi i f_n t} dt \approx \sum_{n=0}^{N-1} h_k e^{2\pi i f_n t_k} \Delta = \Delta \sum_{k=0}^{N-1} h_k e^{2\pi i k n//N}$$

Let

$$H_n \equiv \sum_{k=0}^{N-1} h_k e^{2\pi i k n/N}$$

so $H(f_n) \approx \Delta H_n$ Since $H_{-n} = H_{N-n}$, we can have the discrete Parseval's theorem

$$\sum_{k=0}^{N-1} |h(k)|^2 = \frac{1}{N} \sum_{n=0}^{N-1} |H_n|^2$$