

Given a signal $s(t)$ the goal is to determine the physical parameters of the system that generated the signal. We would like to measure the masses of the two black holes, m_1 and m_2 , although it turns out to be more convenient to work with the total mass $M = m_1 + m_2$ and the “symmetric mass ratio” $\eta = m_1 m_2 / (M^2)$.

We’ll proceed by constructing candidate “template” waveforms and then seeing how well each template matches the signal. Constructing a full template is a really hard problem, and requires numerical relativity. However for our purposes we can use an approximation to the waveform

$$\tilde{h}(f; M, \eta) = \frac{2GM_\odot}{c^2 r} \left(\frac{5M\eta}{96M_\odot} \right)^{\frac{1}{2}} \left(\frac{M}{\pi^2 M_\odot} \right)^{\frac{1}{3}} \left(\frac{GM_\odot}{c^3} \right)^{-\frac{1}{6}} f^{-\frac{7}{6}} e^{i\Psi(f; M, \eta)} \quad (1)$$

where G is Newton’s gravitational constant and c is the speed of light (we usually work in units where both are dimensionless and set to 1), $M_\odot$ is the mass of our sun (which in these units is 4.9×10^{-6} seconds) and r is the distance from the source to Earth. It will turn out that all the terms before $f^{-7/6}$ cancel out and can be ignored, but I included them here for completeness. Note that this gives the Fourier transform of the template!

$\Psi(f; M, \eta)$ is the “phase evolution,” which you can take from equation (3.1) in <http://arxiv.org/pdf/gr-qc/9808076v1.pdf>.

The next question is how to measure how well a template matches the signal. First, define

$$(h | s) = \int_{f_{low}}^{f_{high}} \frac{\tilde{s}(f) \tilde{h}^*(f)}{S_n(f)} df \quad (2)$$

where $S_n(f)$ models the noise in the gravitational-wave detector. This is very much like the vector dot product, where the vectors are elements in the space of functions. Conventionally the low frequency f_{low} is taken to be 40 Hz, and f_{high} can be taken to be 1024 Hz.

Then define the overlap as

$$\langle h | s \rangle = \frac{(h | s)}{\sqrt{(h | h) (s | s)}} \quad (3)$$

This normalizes the “lengths” of the vectors, and so gives a value between 0 and 1. You can also check that this normalization removes all the leading terms in equation (1), leaving only the parts that depend on the frequency.

There are lots of different models for $S_n(f)$, but one good one to use is

$$S_n(x) = (4.49x)^{-56.0} + 0.16x^{-4.52} + 0.52 + 0.32x^2 \quad (4)$$

where $x = f/150.0$.

There is one remaining subtlety. In advance we don’t know the time at which the signal will arrive, and the waveform has an unknown phase which we can think of as the point in the orbit where the signal first becomes visible. One way of addressing this is to add the time shift τ and phase shift ϕ_0 to the set of variables, and do a four-dimensional monte carlo. This is in fact what’s done in real analyses, however, this greatly increases the complexity and time required. Fortunately there are some tricks that give reasonable approximations.

Consider a time-domain signal $h(t)$ with Fourier transform $\tilde{h}(f)$. If we add a time shift τ to the signal then using the definition of the Fourier transform you can show that the transform of $h(t + \tau)$ is $\exp(-2\pi i f \tau) \tilde{h}(f)$. Substituting this into equation (2) gives

$$\int_{f_{low}}^{f_{high}} \frac{\tilde{s}(f) \tilde{h}^*(f)}{S_n(f)} e^{2\pi i f \tau} df \quad (5)$$

but this is just the inverse Fourier transform of the function $\tilde{s}(f)\tilde{h}^*(f)/S_n(f)$, which means we can easily evaluate the overlap for all times in a single operation.

Adding the unknown phase ϕ_0 is even easier, it turns equation (5) into

$$e^{-2\pi i\phi_0} \int_{f_{low}}^{f_{high}} \frac{\tilde{s}(f)\tilde{h}^*(f)}{S_n(f)} e^{2\pi i f \tau} df \quad (6)$$

Because the phase is a constant it just comes out of the integral. Since the magnitude of any number of the form e^{ix} is 1 the phase can then be eliminated by taking the absolute value of (6). We can then choose the best time value by taking the maximum of the resulting series.

Putting it all together, the measure of how well a signal matches a template (which I've informally called the "goodness of fit" and which in proper monte carlo lingo would be called the "likelihood") is

$$L(M, \eta) = \max_{\tau} \frac{1}{\sqrt{(h(M, \eta) | h(M, \eta)) (s | s)}} \left| \int_{f_{low}}^{f_{high}} \frac{\tilde{s}(f)\tilde{h}^*(f)}{S_n(f)} e^{2\pi i f \tau} df \right| \quad (7)$$