Today: 11.1, 11.2.

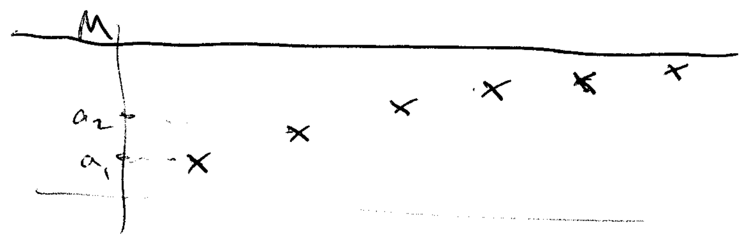
Defn: A sequence $\{a_n\} = \{a_1, a_2, a_3, \dots\}$ is called *increasing* if $a_n \geq a_{n-1}$ for all n.

- - - - *decreasing* if $a_n \leq a_{n-1}$ for all n.

monotone/monotonic is either increasing or decreasing.

A sequence $\{a_n\}$ is *bounded above* if there exists an M s.t. $a_n \leq M$ for all n.

bounded below ...



<u>Thm</u>: Every bonded monotonic sequence converges.
(Theorem)

bounded above and below

$\mathbb{R}$ ← real numbers

$\mathbb{Q}$ ← rational numbers $\left\{ \frac{p}{q} \mid p, q \in \mathbb{Z} \right\}$

$\mathbb{Z}$ ← integers $\{ \dots, -2, -1, 0, 1, 2, \dots \}$

$\pi \in \mathbb{R} - \mathbb{Q}$

$3.1415926\dots$ ← this means I have a sequence

$a_1 = 3$, $a_2 = 3.1$, $a_3 = 3.14$, $a_4 = 3.141, \dots$

π is defined as the limit of this sequence.

(Note that a limit exist because it is increasing and bounded above (e.g. by 4)).

$0, 0.9, 0.99, 0.999, 0.9999, \dots$

limit of this sequence is 1. $0.\overline{9} = 1$

$1, 1.0, 1.00, 1$

1

Series (11.2)

A series is an infinite sum of numbers.

e.g. $3 + 1/10 + 4/10^2 + 1/10^3 + \cdots$ convergent series.

e.g. $1 + 2 + 3 + 4 + \cdots$ divergent series

$$\frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \cdots$$

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$$

Define the <u>partial sums</u> $s_n = \sum_{i=1}^{n} a_i$

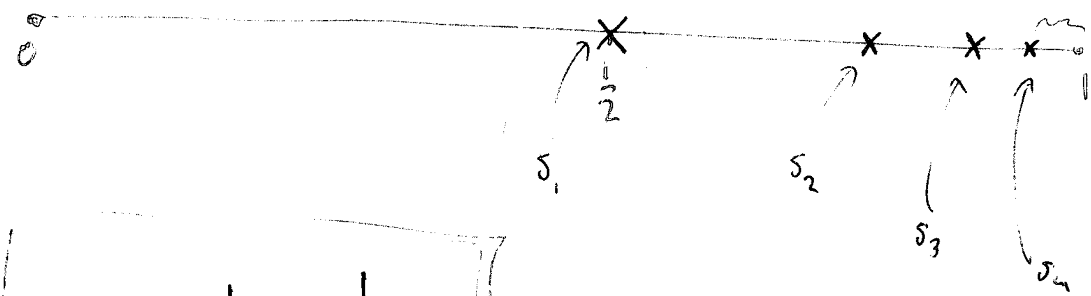
$$= a_1 + a_2 + \cdots + a_n$$

$\{s_n\}$ is a new sequence.

<u>Def-n</u>: We say $\sum_{n=1}^{\infty} a_n = s$ if $\lim_{n\to\infty} s_n = s$.

e.g. $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \cdots = \sum_{n=1}^{\infty} \frac{1}{2^n}$

$$s_n = \sum_{i=1}^{n} \frac{1}{2^i}$$



$$\boxed{s_n = 1 - \frac{1}{2^n}}$$

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \lim_{n\to\infty} (s_n) = \lim_{n\to\infty} \left(1 - \frac{1}{2^n}\right)$$

$$= \underbrace{\lim_{n\to\infty} (1)}_{1} - \cancel{\lim_{n\to\infty}} \left(\frac{1}{2^n}\right)$$

Geometric series:

$$a + a.r + ar^2 + ar^3 + ar^4 + \cdots$$

$$\left(\text{e.g. } a = \tfrac{1}{2},\ r = \tfrac{1}{2}:\ \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots\right)$$

$$S_n = a + ar + \cdots + ar^{n-1}$$

$$\left(S_1 = a,\ S_2 = a + ar,\ S_3 = a + ar + ar^2,\ \ldots\right)$$

$$r.S_n = ar + ar^2 + \cdots + ar^n$$

$$S_n - r.S_n = a + \cancel{ar} + \cdots + \cancel{ar^{n-1}}$$

$$- \left(\cancel{ar} + \cancel{\cdots} + \cancel{ar^{n-1}} + ar^n\right)$$

$$= a - ar^n$$

$$S_n.(1-r) = a.(1-r^n)$$

$$\boxed{S_n = \frac{a.(1-r^n)}{1-r}}$$

If $|r|<1$ $\quad(-1<r<1)$

$$\lim_{n\to\infty}(S_n) = \cancel{\lim} \frac{a}{1-r} \lim_{n\to\infty}\left(1-r^n\right) = \boxed{\frac{a}{1-r}} - \cancel{\frac{a}{1-r}\lim_{n\to\infty}(r^n)}^{\to 0}$$

$$\boxed{\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}}$$

if $|r|<1$

(diverges if $|r| \geq 1$)

$$\sum_{n=0}^{\infty} ar^{n}$$

---

e.g. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

$$s_n = \left(1 - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right) + \cdots + \left(\cancel{\frac{1}{n}} - \frac{1}{n+1}\right)$$

$$s_n = 1 - \frac{1}{n+1} \qquad \lim_{n\to\infty} (s_n) = 1$$