Computational Complexity

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Today

- Polynomial Hierarchy and PSPACE Completeness

Interesting problems in **PSPACE**

- **NP** ⊆ **PSPACE = NSPACE** ⊆ **EXP**

  Savitch's. Thm.

$NP = \{ L \mid$ we have V_L s.t. $x \in L$ iff $\exists w$ $V_L(x, w) = Yes \}$

- More interesting problems in **PSPACE**?

Graph $[G, k] = x$ Clique: is there a subgraph H of G of size $\geq k$.

$x \in L$ iff maximum sub-clique of G has size k. $L \in NP$?

$\exists$ efficient V

$x \in L$ (G, k) iff $\exists w_1 \in \{0,1\}^{q(n)} \forall w_2 \in \{0,1\}^{q(n)}$ $V(x, w_1, w_2)$

$\Sigma_2 = \{ L \mid \exists$ poly-time "verifier" V s.t.

$x \in L$ iff $\left(\exists_{w_1 \in \{0,1\}^{poly(|x|)}} \forall_{w_2 \in \{0,1\}^{poly(|x|)}} V(x, w_1, w_2) = \text{accepts} \right)$ $\}$

$x \notin L$: $\forall w_1 \exists w_2 \; V(x, w_1, w_2)$ reject. V'

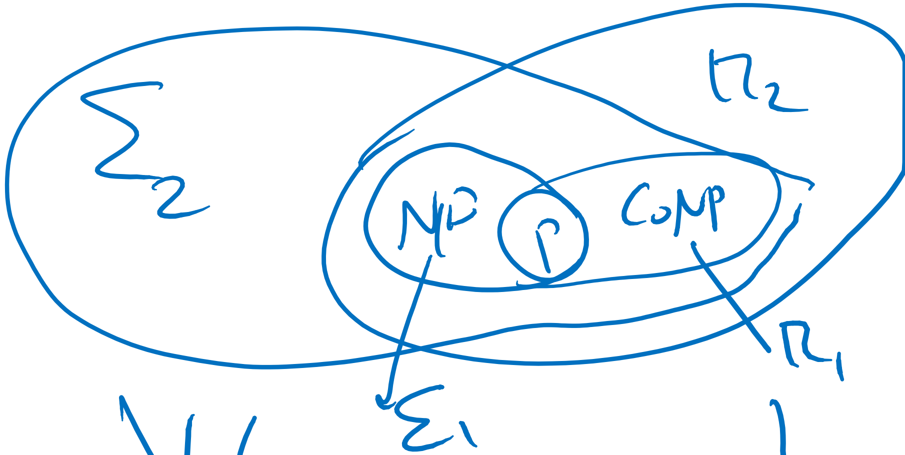
Σ_2 includes:

max-Clique

$NP \subseteq \Sigma_2$

$coNP \subseteq$

$co\text{-}\Sigma_2 = \Pi_2$



$\Pi_2 = \{ L \mid \exists$ poly ... — V ...

$x \in L$ iff $\forall w_1 \exists w_2 \; V(x, w_1, w_2) = \text{yes} \}$

$NP \subseteq \Pi_2$

$co\text{-}N \subseteq$

$x \in L \quad \exists w_1 \exists w_2 \quad M(x, w_1, w_2) = 1$

$\underbrace{w_1 \quad w_2}_{w}$

The i'th level of Hierarchy

Definition 5.3 *(Polynomial hierarchy)*
For $i \geq 1$, a language L is in Σ_i^p if there exists a polynomial-time TM M and a polynomial q such that

$$x \in L \Leftrightarrow \exists u_1 \in \{0, 1\}^{q(|x|)} \forall u_2 \in \{0, 1\}^{q(|x|)} \cdots Q_i u_i \in \{0, 1\}^{q(|x|)} M(x, u_1, \ldots, u_i) = 1$$

where Q_i denotes $\forall$ or $\exists$ depending on whether i is even or odd, respectively.

The *polynomial hierarchy* is the set $\mathbf{PH} = \cup_i \Sigma_i^p$. $= \cup_i \Pi_i$

$\boxed{\Sigma_i \text{ an } \Pi_i \text{ inclue } \Sigma_{i-1}\ \Pi_{i-1}}$

$\Pi_i = co\Sigma_i = \{L \mid \exists$ poly-time M and q $x \in L \Longleftrightarrow \forall u_1 \exists u_2 \cdots M(x, u_1, \ldots u_i) = 1\}$

$\Sigma_2 \subseteq PSPACE$

$L^{\Sigma} \overset{?}{\Longrightarrow} L^{\Sigma}$

$x \in L$ iff $\exists w_1 \forall w_2 \; V(x, w_1, w_2) = 1$

give a $poly(|x|)$ space Algorithm to decide $x \in L$.

given x

For all w_1

good = True.

For all w_2

if $V(x, w_1, w_2) \neq 1$

good = false

break.

if good then return $x \in L$

return $x \notin L$.

$$PH = \begin{matrix} \bigcup_i \Sigma_i \subseteq PSPACE \\ \bigcup_i \Pi_i \subseteq PSPACE \end{matrix}$$

Σ_i has a complete problem (using karp reduction)

NP

coNP.

$\sigma_i = \{ \varphi \mid \exists w_1 \forall w_2 \cdots \varphi(w_1 w_2 \cdots w_i) = 1 \}$

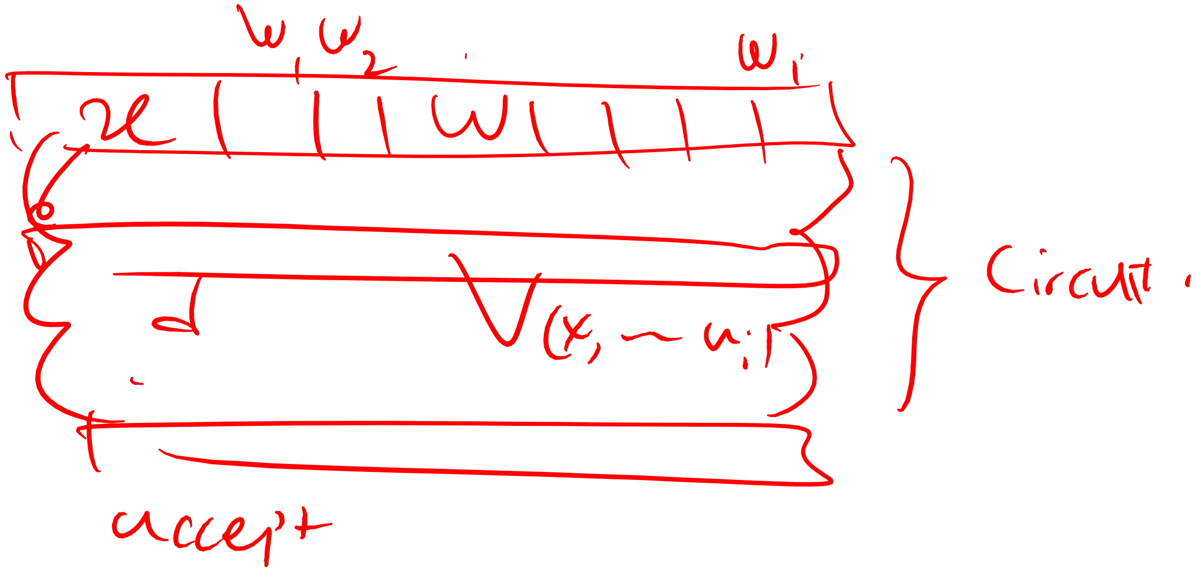
formula.

$\pi_i = \{ \varphi \mid \forall w_1 \exists w_2 \cdots \varphi(w_1 - w_i) = 1 \}$

$\sigma_1 \overset{?}{=} SAT$

σ_i: Complete for Σ_i

π_i: ———— Π_i



What if we allow polynomially many

- True Quantified Boolean Formula (TQBF)

Definition 4.10 *(Quantified Boolean Formula)* A *quantified Boolean formula* (QBF) is a formula of the form $Q_1x_1Q_2x_2 \cdots Q_nx_n\varphi(x_1, x_2, \ldots, x_n)$ where each Q_i is one of the two quantifiers $\forall$ or $\exists$, $x_1, \ldots, x_n$ range over $\{0, 1\}$, and φ is a plain (unquantified) Boolean formula. The quantifiers $\forall$ and $\exists$ have their standard meaning of "for all" and "exists."

Theorem 4.13 *([SM73])*
TQBF *is* **PSPACE**-*complete.*

Definition 4.9 A language L' is **PSPACE**-*hard* if for every $L \in$ **PSPACE**, $L \leq_p L'$. If in addition $L' \in$ **PSPACE** then L' is **PSPACE**-*complete.* $\diamond$