Elementary matrix operations + Systems of linear equations

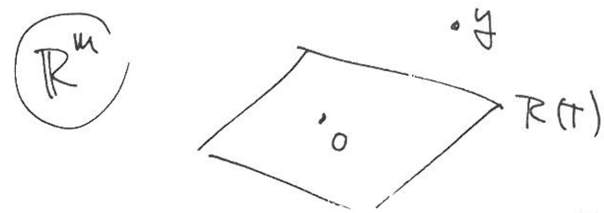
Consider the linear system $Ax = y$, A is $m \times n$, $x \in F^n$, $y \in F^m$

In elementary linear algebra, this is generally regarded as an equation to be solved + in general, there are 3 possibilities:

- No solution – the eq'n is inconsistent (for this y)
- A unique solution
- ∞'ly many solutions

Our current point of view is that $T: \mathbb{R}^n \to \mathbb{R}^m$ given by $Tx = Ax$ $(= L_A x$ in your text$)$ is a linear transformation, and all the above are questions about $R(T)$ – the range of T

- If there's no solution for a given y, then $y \notin R(T)$



Unless $A = 0$, $R(T) \neq \{0\}$, so there are some y's that are $\neq 0$ + in $R(T)$.

- If every $y \in \mathbb{R}^m$ is in $R(T)$, then $R(T) = F^m$ + T is onto.

We have
$$n = \dim \mathbb{R}^n = \dim R(T) + \dim N(T)$$
$$= m + (n-m)$$

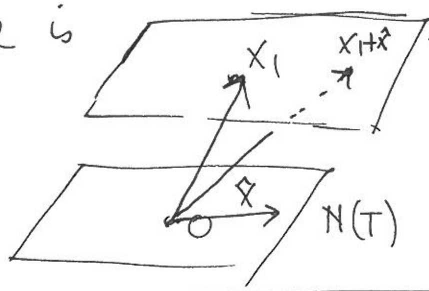
If T is onto and $m = n$, then T is an isomorphism. If $m < n$, then $N(T)$ is not $\{0\}$ and there are non-trivial sol'ns to $Ax = 0$ (the homogeneous eq'n)

Prop: Suppose $Ax = y$ has solutions x_1 + x_2. Then $x_1 - x_2 \in N(T)$.
Equivalently, if x_1 is any sol'n to $Ax = y$, then for any $\hat{x} \in N(T)$, $x_1 + \hat{x}$ is also a sol'n. The general sol'n is the

set $\{X + \hat{X} : \hat{X} \in N(T)\}$.

Pf: $A(X+\hat{X}) = AX + A\hat{X}$ (linearity) $= AX + 0$ $(\hat{X} \in N(T)) = y$.

The picture is $\mathbb{R}^n$



← set of all sol'ns to $Ax=y$

This is not a subspace, unless $y=0$. Why?

Commutative Diagrams: Suppose $T: V \to V$ is linear + $\dim V = n$. Let E be a basis for V. We then have

a) $V \to V_E \to AV_E$, and

b) $V \to TV \to (TV)_E$. We've shown $(TV)_E = AV_E$. So there are 2 different ways to get from V to $(TV)_E$:

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "n_E"'] & V \arrow[d, "n_E"] \\ \mathbb{R}^n \arrow[r, "A"] & \mathbb{R}^n \end{tikzcd}$$

Here n_E is the isomorphism that sends $v \in V$ to v_E, the coords of v in the basis E. We call this figure a "commutative diagram": you can get from the upper left (V) to the lower right ($\mathbb{R}^n$) by 2 paths $\urcorner$ + $\llcorner$, and both give the same result

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In this particular case, since n_E is an isomorphism, we can "reverse" things:

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "A"] \arrow[d, "n_E^{-1}"'] & \mathbb{R}^n \arrow[d, "n_E^{-1}"] \\ V \arrow[r, "T"] & V \end{tikzcd}$$

Starting with v_E in the upper left, we get to Tv in 2 different ways as well. These diagrams are convenient mnemonic devices for keeping track of what's going on.

More generally, if $T: V \to W$, and E & H are bases in V, W respectively, we have the diagram

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "n_E"'] & W \arrow[d, "n_H"] \\ \mathbb{R}^n \arrow[r, "A"] & \mathbb{R}^m \end{tikzcd}$$

where A is the mx of T in the 2 bases & n_E & n_H are the isomorphisms assigning local coordinates.

Because of these diagrams, we can study T by studying A and vice versa.

Elementary row & column operations:

There are 3:

1. replacing row i by row i + $c \cdot$ row j (or col i by col i + c col j
2. replacing row i by $c \cdot$ row i (col i by $c \cdot$ col i)
3. interchanging 2 rows (columns)

If A is $m \times n$, then the row operations are done by

a) Doing the row operation to the $m \times m$ identity mx I_m to obtain the elementary row matrix P.

b) Forming the product PA

The column operations are done by a) doing the column operation to the $\;x$ identity I & b) forming the product AQ.

Note: For any $m \times m$ matrix P, invertible or not, $\overleftrightarrow{PA}$ has rows which are linear combinations of the rows of A.
Similarly, AQ consist of cols gotten from A by linear combs

Example: $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}_{2\times 3}$

a) replace row 2 by row 2 + 3 · row 1

$$P = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}, \quad PA = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 7 & 11 & 15 \end{pmatrix}$$

b) replace col 2 by col 2 + 3 col 1

$$Q = \begin{pmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad AQ = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}\begin{pmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 5 & 3 \\ 4 & 17 & 6 \end{pmatrix}$$

Prop: All e.r.o. & e.c.o.'s regarded as linear transformations are invertible

Pf: The inverses are given by "undoing" the operation e.g. To "undo" $P = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}$, we replace row 2 by row 2 − 3 row 1, giving the elem mx

$$R = \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix}, \quad \text{and } PR = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= RP \Rightarrow R = P^{-1}$$

Defn: The rank of the $m \times n$ matrix A is the rank of any linear transformation T corresponding to it. That is $\text{rank}(A) = \dim R(T)$ for any corresponding T.

Theorem: Let P & Q be invertible $m\times m$ & $n \times n$ matrices & let A be $m\times n$. Then

$$\text{rank}(PAQ) = \text{rank}(A)$$

Pf: 1. Consider $PA: \mathbb{R}^n \to \mathbb{R}^m$

$$A: \mathbb{R}^n \longrightarrow R(A) \subseteq \mathbb{R}^m$$

P is an isomorphism, so $\dim P(R(A)) = \dim(R(A))$

& $\therefore$ rank PA = rank A

2. $AQ: \mathbb{R}^n \to \mathbb{R}^m$. Q is an isomorphism & so

$$Q(\mathbb{R}^n) = \mathbb{R}^n$$

$$\therefore (AQ)(\mathbb{R}^n) = A(\mathbb{R}^n) = R(A)$$

& so rank AQ = rank A

3. Combining 2 & 3, rank (PAQ) = rank (A)

Cor: Row & column operations on A leave rank(A) invariant

Cor: Using row operations to convert A into Gauss-Jordan or reduced echelon form does not change the rank.

Remark: Recall that reduction goes like

$$\underbrace{E_k \cdots E_2 E_1}_{k \text{ elem. row operations}} A = R \quad \nwarrow \text{ GJ form of } A$$

Now R has the form (for example)

$$\underbrace{\begin{pmatrix} 1 & * & 0 & * & 0 & * \\ 0 & 0 & 1 & * & \vdots & * \\ \vdots & \vdots & \vdots & & 1 & * \\ 0 & 0 & 0 & & 0 & \end{pmatrix}_{m \times n}}_{l \text{ leading 1's}}$$

We can now use column operations to use the leading 1's to zero out the rest of the corresp. rows - e.g.

$$\begin{pmatrix} 1 & 0 & \cdots & & & & 0 \\ 0 & 0 & 1 & 0 & \cdots & & 0 \\ \vdots & & \vdots & & & & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \cdots & 0 \end{pmatrix}$$

& then permute the columns to get all the leading 1's adjacent

resulting in
$$\left(\begin{array}{cccc|c} 1 & 0 & \cdots & 0 & \\ 0 & 1 & & & O_{\ell\times n-\ell} \\ \vdots & & \ddots & & \\ 0 & 0 & \cdots & 1_\ell & \\ \hline & O_{m-\ell\times\ell} & & & O_{m-\ell\times n-\ell} \end{array}\right)$$
Here the upper left hand block is $\ell\times\ell$.

Cor: If A is reduced to GJ form R, then rank (A) $=$ # leading 1's in R.

From the above, the dimension theorem $\Rightarrow$

cols of $A = n = \text{rank}(A) + \text{nullity}(A)$

So if $\text{rank}(A) = \ell$, then ~~and~~ $\text{nullity}(A) = n-\ell$.

Remarks: ① range (PA) is not the same as range (A) in general although they have the same dimension.

② If A is $m\times n$, then $\text{rank}(A) \le \min\{m,n\}$

Now finally, consider the question of solving $Ax=y$. We do row operations on the augmented matrix $(A \,\vdots\, y)$ and reduce this to GJ form.

a) If $Ax=y$, then doing a row operation results in $PAx=Py$. Since P is invertible, $PAx=Py \Leftrightarrow Ax=y$

b) So x is a sol'n to $Ax=y \Leftrightarrow x$ is a sol'n to $PAx=Py$

c) We keep going until we can answer the question

d) If $y=0$ (homogeneous), then there's always one sol'n, namely $x=0$. This is called the trivial sol'n. There are non-trivial sol'ns $\Leftrightarrow N(T)\neq\{0\}$.

e) If $y\neq 0$, then sol'ns exist $\Leftrightarrow$ row reduction of $(A\,\vdots\, y)$ leads to a matrix with no leading 1's in the last column.