Review sheet for Exam 1 - Math 590

DEFINITIONS: vector space, subspace, the vector space $V+W$, where V and W are both subspaces of some larger space X, $X = V \oplus W$ (direct sum), linear combination, span of a set of vectors, linear dependence and independence, basis, dimension, linear transformation, N(T), R(T), rank and nullity of T, the matrix of a linear transformation $T : V \to W$ with respect to two ordered bases in V, W, the coordinates of a vector v in a given basis, isomorphism of vector spaces, inverse of a linear transformation (if it exists), elementary row and column operations and the associated matrices, reduced echelon or Gauss-Jordan form of a matrix.

THINGS YOU SHOULD KNOW: Here the word "know" means either being able to prove the result or use it constructively, depending on the problem. If the proof is fairly long and involved, as in (2) or (6) below, you should be able to give a rough sketch of why the result is true.

1. The answers to many questions in linear algebra depend on whether or not certain systems of linear equations have solutions. What computation would you do in the following cases?
   (a) Do the vectors in a set $S = \{e_1, e_2, \ldots, e_k\} \subset V$ span the space V?
   (b) If so, do they form a basis for V?
   (c) Does $y \in W$ belong to the range of T, where $T : V \to W$?
   (d) Is the linear transformation $T : V \to W$ 1-1? Onto?
   (e) What is the rank of T?
2. The dimension theorem: If $T : V \to W$ is linear, and V is finite-dimensional, then $\dim(V) = \dim R(T) + \dim N(T)$.
3. If $T : V \to W$ is linear and both spaces are finite dimensional, with $E = \{e_1, \ldots, e_n\}$ and $F = \{f_1, \ldots, f_m\}$ bases for V and W respectively,
   (a) Find the coordinates of the vector $v \in V$ in the basis E
   (b) Find the matrix representation of T in the bases E and F.
   (c) Find the coordinates of Tv in the basis F.
   (d) Write down a commutative diagram summarizing the 3 items above.
4. If $\dim(V) = \dim(W)$ (finite), the following are equivalent: (1) T is 1-1, (2) T is onto, $\text{rank}(T) = \dim(V)$.
5. Let A be an $m \times n$ matrix, P an invertible $m \times m$ matrix, and Q an invertible $n \times n$ matrix. Then $\text{rank}(PAQ) = \text{rank}(A)$.

6. Given $A_{m\times n}$, $\exists P_{m\times m}, Q_{n\times n}$, both invertible, such that
$$PAQ = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix},$$
where I is $k \times k$, with $k =$rank(A). What are the dimensions of the three 0 matrices?

7. When we change bases in V and W, according to $E = FP$ and $G = HQ$, then if A is the matrix of T in the bases E, G, the matrix in the new bases is $B = P^{-1}AQ$. It follows from this and the results above that the rank of any matrix representing a linear transformation T is well-defined (i.e, independent of the particular bases chosen. Why?

8. Give another statement of the dimension theorem in terms of a matrix representing a linear transformation. What do the leading and free variables have to do with this?

Computations and simple proofs: (these are just examples)

1. Show that $\{1, x+1, x^2-2\}$ is a basis for the vector space $P_2(\mathbb{R})$.

2. If A is 2×3, then $\dim(N(L_A)) > 0$.

3. Show that if $\{e_1, \ldots, e_n\}$ and $\{f_1, \ldots, f_m\}$ are both bases for the vector space V, then $n = m$. (i.e., the notion of dimension is well-defined).

4. Two vector spaces of the same finite dimension are isomorphic,

5. The number of leading variables in the matrix A (after row reduction) is equal to its rank.

6. Suppose there's a unique solution x to $Ax = y$ for some y. If there's a solution x_1 to $Ax_1 = y_1 (\neq y)$, is this solution unique as well? If $Ax = y$ has a unique solution is there a solution for any y_1?

7. If $T : V \to W$ is linear, then T is 1-1 $\iff N(T) = \{0\}$. What is the rank of T in this case?

8. Let W_1 be the subset of $P_6(\mathbb{R})$ consisting of all polynomials of degree ≤ 6 whose terms are all of even degree together with the 0 polynomial. Show that W_1 is a subspace. What is its dimension?

   If W_2 is the set of polynomials in $P_6(\mathbb{R})$ whos terms all have odd degree, together with the 0 polynomial, show that $P_6(\mathbb{R}) = W_1 \oplus W_2$.

9. Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by $T(x, y, z) = (x + 2y + z, y - z)$. What is the matrix representative of T in the standard bases in $\mathbb{R}^3$ and $\mathbb{R}^2$?

10. If $\{e_1, \ldots, e_n\}$ is a basis for V, and $v \in V$, show that the coordinates v_E of v in the basis E are unique.

To give a "rough sketch" of a proof means to present the main ideas of the proof in order, without going into details. Here's an example of the most difficult proof we've done:

Sketch of a proof of the dimension theorem: We have $T : V \to W$, V finite-dimensional. Suppose $\dim(V) = n$. Choose a basis $\{f_1, f_2, \ldots, f_k\}$ for $R(T) \subseteq W$, where $\dim(R(T)) = k \leq n$. Now choose vectors $\{e_1, \ldots, e_k\}$ in V such that $T(e_i) = f_i$. The vectors $\{e_i : 1 \leq i \leq k\}$ are linearly independent. Now choose $n - k$ additional vectors to form a basis $\{e_1, \ldots, e_k, e_{k+1}, \ldots e_n\}$ of V. For each m, $k < m \leq n$, $T(e_m) \in R(T)$. It follows that $T(e_m) = \sum_{i=1}^k c_i f_i$ (because $\{f_1, \ldots, f_k\}$ form a basis for $R(T)$). By linearity of T, this means that $T(e_m - \sum_{i=1}^k c_i e_i) = 0$, so $g_m := e_m - \sum_{i=1}^k c_i e_i \in N(T)$. It can be shown that the $g_m : k < m \leq n$ are linearly independent. So $\{e_1, \ldots, e_k, g_{k+1}, \ldots, g_m\}$ forms a basis for V. Finally, if $v \in V$, then $v = c_1 e_1 + \cdots c_k e_k + c_{k+1} g_{k+1} + \cdots + c_n g_n$, so $T(v) = c_1 f_1 + \cdots + c_k f_k$ (because all $g_m \in N(T)$). Since the f_i are linearly independent, $T(v) = 0 \iff c_1 = \cdots c_k = 0 \iff v \in \text{ span}(g_{k+1}, \ldots, g_n)$. So the set $\{g_{k+1}, \ldots, g_n\}$ is a basis for $N(T)$ which therefore has dimension $n - k$.

A number of details have been omitted (e.g., proof that the $e_i : 1 \leq i \leq k$ are linearly independent), but there's enough information here to construct a full proof.

Your text gives another proof, starting with the assumption that $\dim(N(T))$ is given and deducing the dimension of $R(T)$.