Math 590 - Exam 1 Solutions

DIRECTIONS: Show all your work on these papers. Use the back if you run out of space. If you're running out of time, try to indicate what should be done on the uncompleted problems. For instance, just leave Ax as it is, instead of multiplying it out.

I: DEFINTIONS:

1. The set of vectors $\{e_1, e_2, \ldots, e_k\} \subset V$ is *linearly dependent* if $\exists c_1, \ldots, c_k,$ not all 0, such that $\sum c_i e_i = 0$.
2. The set of vectors $\{e_1, e_2, \ldots, e_k\} \subset V$ forms a *basis* for V if the set is linearly independent and spans V.
3. If V and W are vector spaces, a function $T : V \to W$ is a *linear transformation* if $T(cu + v) = cT(u) + T(v)\ \forall u, v \in V, c \in F$.
4. The linear transformation $T : V \to W$ is an *isomorphism* if it's 1-1 and onto. Equivalently, $\exists U : W \to V$ such that $UT = I_V, TU = I_W$.
5. The *span* of the set $\{e_1, e_2, \ldots, e_k\} \subset V$ is the set of all linear combinations of elements of the set.

II: Mostly computation

1. If $T : V \to W$ is linear, show that $R(T)$ is a subspace of W.

   Let $y_1, y_2 \in R(T), c \in F$. We need to show that $cy_1+y_2 \in R(T)$. $\exists x_1, x_2 \in V$ such that $Tx_i = y_i$ (because the $y's$ are in the range). Therefore $cy_1 + y_2 = cTx_1 + Tx_2 = T(cx_1 + x_2)$ (linearity of T). Done...

2. Let $T : P_2(\mathbb{R}) \to P_3(\mathbb{R})$ be given by $(Tf)(x) = x^3 f''(x) - 2xf'(x) + f(x)$. You may assume T is linear. What is the rank of T? What is the nullity of T?

   We have $T1 = 1;\ Tx = -x;\ Tx^2 = 2x^3 - 3x^2$. These are linearly independent (look at the degrees) so rank(T) = 3. By the dimension theorem, nullity(T) = 0.

3. Let
$$e_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \quad e_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$
   be a basis for $\mathbb{R}^2$. Find the coordinates of $v = (3, 4)$ in this basis.

   If $v = c_1e_1 + c_2e_2$, then we get the equations $c_1 - c_2 = 3;\ -2c_1 + c_2 = 4$. The solutions are $c_1 = -7,\ c_2 = -10$.

   If $f_1 = 3e_1 - e_2,\ f_2 = e_1 + 2e_2$, find the coordinates of the same vector v in the basis $\{f_1, f_2\}$.

   We have $F = EP$, where
$$P = \begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix}, \quad \text{and } v_F = P^{-1}v_E = (1/7)\begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} -7 \\ -10 \end{pmatrix} = \frac{-1}{7}\begin{pmatrix} 4 \\ 37 \end{pmatrix}$$

4. Let $\{e_1, e_2\}, \{f_1, f_2\}$ be the same as in problem (3) above. Suppose the linear tranformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by

$$Te_1 = e_1 + 2e_2,\ Te_2 = 2e_1 + e_2,$$

and extending by linearity. Find the matrix representation A of T in the basis $\{e_1, e_2\}$. Then find the matrix representation of T in the basis $\{f_1, f_2\}$.

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix};\ B = P^{-1}AP = (1/7)\begin{pmatrix} -3 & 6 \\ 16 & 17 \end{pmatrix}$$

III: (True/False): You *must* give a reason for your answer!

1. If $T : V \to W$ is linear and 1-1, then T is onto.

   False: for instance, $(x, y) \to (x, y, 0)$ is 1-1 but not onto, and it's linear.

2. If $\dim(V) = \dim(W) = k$ (finite) and $T : V \to W$ is onto, then T is 1-1.

   True: if T is onto, then rank(T) = k, so dim(N(T)) = 0 and therefore T is 1-1.

3. The range of the linear transformation $L_A : \mathbb{R}^n \to \mathbb{R}^m$ is the space generated by the columns of the matrix A.

   True: If $y = Ax$, so y is in the range, then $y = \sum_i x_i \text{col}_i(A)$.

4. If the Gauss-Jordan form of the matrix A has a row of zeros, then the null space of A has dimension ≥ 1.

   False: for example

   $$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

   is in Gauss-Jordan form and has $N(T) = \{0\}$.

IV: Do *one* of the following:

1. Suppose $T : V \to W$ is linear and $\{f_1, f_2, \ldots, f_k\}$ is a basis for $R(T)$. Let $\{e_1, e_2, \ldots, e_k\}$ be vectors in V such that $Te_i = f_1$, for $1 \leq i \leq k$.

   (a) Show that the set $\{e_1, e_2, \ldots, e_k\}$ is linearly independent.
   If $\sum_i c_i e_i = 0$, then $T(\sum_i c_i e_i) = \sum_i c_i f_i = 0 \Rightarrow c_1 = \cdots = c_k = 0$ (because the f's are lin. ind). So this means all the $c's$ vanish and the e's are lin ind.

   (b) Suppose there exists a vector $v \in V$, which is *not* in the span$\{e_1, \ldots, e_k\}$. Show that there exist constants $c_1, \ldots, c_k$ such that $v - \sum_{i=1}^k c_i e_i$ belongs to $N(T)$.
   Given v, we must have $Tv = R(T) \Rightarrow Tv = \sum_i c_i f_i$ for some coefficients c_i. Then $Tv = T(\sum_i c_i e_i) \Rightarrow T(v - \sum_i c_i e_i) = 0 \Rightarrow v - \sum_i c_i e_i \in N(T)$.

2. Suppose that $\{e_1, \ldots, e_n\}$ and $\{f_1, \ldots, f_m\}$ are both bases for the vector space V. Show that $m = n$.

   Suppose not, and that $m < n$. We'll show that $\{e_1, e_2, \ldots, e_n\}$ must be linearly dependent and therefore not a basis. Similar reasoning will apply if $n < m$ leaving $m = n$ as the only possibility.

   Since the f_i are a basis, we may write $e_1 = a_1 f_1 + \cdots + a_m f_m$ for some coefficients a_i, not all of which are 0. By renumbering if necessary, we may suppose that $a_1 \neq 0$. Then we can solve algebraically for f_1 as a linear combination of $\{e_1, f_2, f_3, ..., f_m\}$. This set, which contains m vectors, is a basis: f_1 is a lin. comb. of all these, so the span of this set includes all the f_i, so it spans V. And it's linearly independent, since any lin. comb. of $e_1, f_2, ...f_m$ can be converted into a lin. comb. of $f_1, f_2, ..., f_m$ and therefore vanishes $\iff$ all coefficients vanish.

   We then repeat the argument, replacing f_2 (possibly renumbered) with e_2, again obtaining a basis for V. Continue along until we have the set of m vectors $\{e_1, e_2, \ldots, e_m\}$ as a basis for V. It then follows that $e_{m+1}, \ldots, e_n$ are in the span of these m vectors, and so the n vectors are linearly dependent, hence not a basis, as claimed.

3. If A is $m \times n$, and $P_{m\times m}$ and $Q_{n\times n}$ are both invertible, then $\text{rank}(PAQ) = \text{rank}(A)$.

   We consider the linear transformation $L_A : F^n \to F^m$. First, look at AQ or $L_A L_Q$. L_Q is an isomorphism of F^n, so $L_Q(F^n) = F^n$ (as sets). Therefore, $\text{rank}(L_{AQ}) = \text{rank}(L_A) \Rightarrow \text{rank}(AQ) = \text{rank}(A)$.

   Next consider PA or $L_P L_A$. Since L_P is an isomorphism it maps the range of A onto a set of the same dimension (even though it doesn't preserve the range). So $\text{rank}(PA) = \text{rank}(A)$. Combining these two gives the result.

Remark: There are undoubtedly other ways to do these problems....