Select Solutions

3. Determine the radius of convergence of the following power series.

(ii) (**10 pts.**) $\sum\limits_{n=1}^{\infty} \frac{1^2 \cdot 3^2 \cdot 5^2 \dots (2n-1)^2}{2^2 \cdot 4^2 \cdot 6^2 \dots (2n)^2} x^{2n}$.

Solution: We apply the generalized ratio test to find the radius of convergence:

$$\lim_{n\to\infty} \frac{\frac{1^2 \cdot 3^2 \cdot 5^2 \dots (2n-1)^2 (2n+1)^2}{2^2 \cdot 4^2 \cdot 6^2 \dots (2n)^2 (2n+2)^2} |x|^{2n+2}}{\frac{1^2 \cdot 3^2 \cdot 5^2 \dots (2n-1)^2}{2^2 \cdot 4^2 \cdot 6^2 \dots (2n)^2} |x|^{2n}} = \lim_{n\to\infty} \frac{(2n+1)^2 |x|^2}{(2n+2)^2} = |x|^2 < 1 = R$$

as we have

$$\lim_{n\to\infty} \frac{(2n+1)^2}{(2n+2)^2} = \lim_{n\to\infty} \frac{(2+1/n)^2}{(2+2/n)^2} = 1.$$

(iii) (**10 pts.**)For what values of x does $\sum\limits_{n=1}^{\infty} nx^n$ converge? Show that for the x values that you found, $\sum\limits_{n=1}^{\infty} nx^n = \frac{x}{(1-x)^2}$.

Solution: Using the root test we have that

$$\lim_{n\to\infty} |nx^n|^{1/n} = \lim_{n\to\infty} n^{1/n}|x| = |x| < 1$$

so it converges for $|x| < 1$. Plugging in 1 and -1 shows it does not converge for those values. To show $\sum\limits_{n=1}^{\infty} nx^n = \frac{x}{(1-x)^2}$, note that for $|x| < 1$ the geometric formula tells us that

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}.$$

If we differentiate both sides we have

$$\sum_{n=1}^{\infty} nx^{n-1} = \frac{1}{(1-x)^2}.$$

Finally, multiplying both sides by x yields the desired result:

$$\sum_{n=1}^{\infty} nx^n = \frac{x}{(1-x)^2}.$$

4. (i) (**10 pts.**) Find the Taylor expansion of $f(x) = \cos^2(x)$. (**Hint**: Use the trigonometric identity $\cos^2 x = \frac{1}{2}(1+\cos 2x)$)

Solution: Using the fact that $\cos y = \sum_{j=0}^{\infty} \frac{(-1)^n y^{2n}}{(2n)!}$, the hint, and the substitution $y = 2x$, we have that

$$\cos^2 x = \frac{1}{2}(1+\cos 2x) = \frac{1}{2}(1 + \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n}}{(2n)!}).$$

(ii) (**10 pts.**) Use the **Lagrange Remainder formula** (i.e. the remainder formula for Taylor series) to prove that the series converges for all values of x.

Solution: The Lagrange Remainder formula is given by

$$r_n(x) = |\cos^2(x) - \frac{1}{2}(1 + \sum_{j=0}^{n} \frac{(-1)^j (2x)^{2j}}{(2j)!})|.$$

We have that our power series is correct for all x if and only if $\lim_{n\to\infty} r_n(x) = 0$ for all x. To demonstrate this we have the following:

$$\begin{aligned}
\lim_{n\to\infty} r_n(x) &= \lim_{n\to\infty} |\cos^2(x) - \frac{1}{2}(1 + \sum_{j=0}^{n} \frac{(-1)^j (2x)^{2j}}{(2j)!})| \\
\text{(by the hint)} \qquad &= \lim_{n\to\infty} |\frac{1}{2}(1+\cos 2x) - \frac{1}{2}(1 + \sum_{j=0}^{n} \frac{(-1)^j (2x)^{2j}}{(2j)!})| \\
&= \lim_{n\to\infty} |\frac{1}{2}\cos 2x - \frac{1}{2} \sum_{j=0}^{n} \frac{(-1)^j (2x)^{2j}}{(2j)!}| \\
(y = 2x) \qquad &= \lim_{n\to\infty} \frac{1}{2} |\cos y - \sum_{j=0}^{n} \frac{(-1)^j (y)^{2j}}{(2j)!}| \\
&= 0
\end{aligned}$$

where the last equality is due to the fact that $\cos y = \sum_{j=0}^{\infty} \frac{(-1)^j y^{2j}}{(2j)!}$ for all y.