MATH 141 Exam 4 Solutions

1. (i)(**15 pts**) Find all z so that

$$\frac{z^2}{z^2+i} = -1.$$

SOLUTION: After multiplying both sides by z^2+i, we have

$$z^2 = -z^2 - i \iff 2z^2 = -i \iff z^2 = -\frac{i}{2}.$$

Since $-i = \cos(3\pi/2) + i\sin(3\pi/2) = e^{i3\pi/2}$ and $|\frac{-i}{2}| = 2^{-1}$, we have

$$z = 2^{-1/2}e^{i3\pi/4} \text{ and } z = 2^{-1/2}e^{i7\pi/4}.$$

(ii) (**15 pts**) Find 5 distinct z such that $z^7 = -i$.
SOLUTION: Since $-i$ is on the unit circle we know that all of its roots are also on the unit circle. Hence the modulus of z will also be one (i.e. $z = e^{i\theta}$). From the previous problem, we know $-i = e^{i3\pi/2}$ so we have

$$z^7 = -i \iff e^{i\theta 7} = e^{i3\pi/2} \Rightarrow \theta = 3\pi/14.$$

Now for $k = 0, 1, ..., 6$ we also have $-i = e^{i(3\pi/2+k2\pi)}$, so by the same method we have

$$z_k = e^{i(3\pi/14+k2\pi/7)}.$$

Any subset of five of these is sufficient.
2. For (i), (ii), and (iii) determine whether the series diverges, converges conditionally, or converges absolutely.
(i) (**5 pts.**) $\sum\limits_{n=1}^{\infty} \frac{\sin n\pi}{n+1}$
SOLUTION: For all natural numbers n, $\sin n\pi = 0$ so the sum converges to zero absolutely.
(ii) (**5 pts.**) $\sum\limits_{n=2}^{\infty} \frac{(-1)^n}{(n^5-1)^{1/4}}$
SOLUTION: Using the generalized limit comparison test with $b_n = 1/n^{5/4}$, we have

$$\lim_{n\to\infty} \frac{\frac{1}{(n^5-1)^{1/4}}}{\frac{1}{n^{5/4}}} = \lim_{n\to\infty} \frac{n^{5/4}}{(n^5-1)^{1/4}} = \lim_{n\to\infty} \frac{1}{\frac{n^{5/4}}{n^{5/4}}(1-\frac{1}{n^5})^{1/4}} = 1 > 0.$$

Since $\sum_{n=2}^{\infty} \frac{1}{n^{5/4}}$ converges, we know by the limit comparison test that $\sum\limits_{n=2}^{\infty} \frac{(-1)^n}{(n^5-1)^{1/4}}$ converges absolutely.
(iii) (**10 pts.**) $\sum\limits_{n=1}^{\infty} (-1)^n \frac{n!}{n^n}$.
SOLUTION: Using the generalized ratio test we have

$$\lim_{n\to\infty} \frac{\frac{(n+1)!}{(n+1)^{n+1}}}{\frac{n!}{n^n}} = \lim_{n\to\infty} \frac{(n+1)!n^n}{n!(n+1)^{n+1}} = \lim_{n\to\infty} \frac{n+1}{n+1}\left(\frac{n}{n+1}\right)^n = \frac{1}{e} < 1.$$

Hence, we have absolute convergence by the generalized ratio test.

(iv) (**5 pts.**) Find the sum of the series $\sum_{n=0}^{\infty} \frac{4^n+5^n}{4^n 5^n}$.

SOLUTION: We have

$$\sum_{n=0}^{\infty} \frac{4^n+5^n}{4^n 5^n} = \sum_{n=0}^{\infty} \frac{4^n}{4^n 5^n} + \sum_{n=0}^{\infty} \frac{5^n}{4^n 5^n} = \sum_{n=0}^{\infty} \frac{1}{5^n} + \sum_{n=0}^{\infty} \frac{1}{4^n} = \frac{1}{1-\frac{1}{5}} + \frac{1}{1-\frac{1}{4}} = \frac{5}{4} + \frac{4}{3},$$

where the last equality comes from the geometric series formula.

3. Determine the radius of convergence of the following power series.

(i) (**5 pts.**) $\sum_{n=1}^{\infty} \frac{5x^{2n}}{3^n n^2}$

SOLUTION: Using the generalized root test we have

$$\lim_{n\to\infty} \left|\frac{5x^{2n}}{3^n n^2}\right|^{1/n} = \lim_{n\to\infty} \left|\frac{5^{1/n}x^2}{3n^{2/n}}\right| = \frac{1}{3}|x|^2 < 1 \iff |x| < \sqrt{3} = R.$$

(ii) (**10 pts.**) $\sum_{n=1}^{\infty} x^{(n^2)}$.

Using the generalized root test, we have

$$\lim_{n\to\infty} |x|^{n^2/n} = |x|^n = \begin{cases} 0 & |x| < 1 \\ 1 & |x| = 1 \\ \text{diverges} & |x| > 1 \end{cases}.$$

Therefore, the radius of convergence is $R = 1$.

(iii) (**10 pts.**)For what values of x does $\sum_{n=0}^{\infty} x^n$ converge? Use the geometric series formula to show what function it converges to. Show that for the x values that you found, $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = \tan^{-1}(x)$.

SOLUTION: For $|x| < 1$, we have

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

by the geometric series formula. Since $\frac{d}{dx}\tan^{-1} t = \frac{1}{1+t^2}$, substituting $t^2 = -x$ in $\sum_{n=0}^{\infty} x^n$, and integrating yields

$$\sum_{n=0}^{\infty} (-1)^n \frac{t^{2n+1}}{2n+1} = \tan^{-1}(t)$$

as desired.

4. (i) (**10 pts.**) Find the Taylor expansion of $f(x) = \cos(x^2)$.

SOLUTION: Using the power series $\cos(y) = \sum_{n=0}^{\infty} (-1)^n \frac{y^{2n}}{(2n)!}$, and the substitution $y = x^2$ yields

$$\cos(x^2) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n}}{(2n)!}.$$

(ii) (**10 pts.**) Use the **Lagrange Remainder formula** (i.e. the remainder formula for Taylor series) to prove that the series converges for all values of x.

SOLUTION: The Lagrange Remainder formula for a function f is given by $r_j(x) = \left|\frac{f^{(j+1)}(t_x)}{(j+1)!}x^{j+1}\right| = \left|f(x) - \sum_{n=0}^{j} \frac{f^{(n)}(x_0)}{n!}x^n\right|$, for some unknown t_x on the interval $[x_0, x]$. For $f(x) = \cos(x^2)$ we have

$$r_j(x) = \left|\cos(x^2) - \sum_{n=0}^{j}(-1)^n \frac{x^{4n}}{(2n)!}\right| = \left|\cos(y) - \sum_{n=0}^{j}(-1)^n \frac{y^{2n}}{(2n)!}\right| \to 0$$

as $j \to \infty$. Therefore we have the correct power series.