MATH 141 Exam 3 (Friday, April 4, 2014)

Please answer each numbered question on a different answer sheet. Make sure your name, your TAs name, your section number, and the appropriate problem number appear on each answer sheet. You only need to sign the pledge on the first answer sheet. Show all work and be sure your answers are complete, with explanations where necessary. You are not allowed any outside materials other than a writing utensil. Please make sure your cell phones are OFF! Good luck!

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1. Evaluate the following:

$$\text{(a) (15 pts)} \int_0^{\pi/2} \tan^4(x/2)\sec^4(x/2)dx$$

We use the substitution $u = \tan(x/2)$ so that $2du = \sec^2(x/2)$. Then $\sec^4(x/2) = \sec^2(x/2)\sec^2(x/2) = \sec^2(x/2)(\tan^2(x/2)+1)$ and the new bounds are 0 to 1. So the new integral is

$$2\int_0^1 u^4(u^2+1)du = \frac{2}{7} + \frac{2}{5} = \frac{24}{35}.$$

$$\text{(b) (15 pts)} \int_0^{\pi} \sin^5(x)\cos^8(x)dx$$

Here we let $u = \cos(x)$ so $-du = \sin(x)dx$. Then the bounds become 1 to -1 (we can take the negative from du to switch the order to -1 to 1). Also $\sin^5(x) = \sin(x)(1-\cos^2(x))^2 = \sin(x)(1-2\cos(x)+\cos^2(x))$. So the new integral is

$$\int_{-1}^1 (u^8 - 2u^{10} + u^{12})du = \frac{1}{9} - \frac{2}{11} + \frac{1}{13} - (-\frac{1}{9} + \frac{2}{11} - \frac{1}{13}) = \frac{2}{9} - \frac{4}{11} + \frac{2}{13}.$$

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2. Consider the function $g(x) = \ln(x+2)$.

(a) (10 pts) Approximate $\int_{-1}^1 g(x)dx$ using the Simpson's rule with 2 subintervals (You should use log rules to simplify your response to a single term).

$$\frac{1--1}{3(2)}(\ln(1) + 4\ln(2) + 1\ln(3)) = \frac{1}{3}(\ln(16) + \ln(3)) = \frac{\ln(48)}{3}$$

Note: Even with just 2 intervals this is accurate to 3 decimal places!

(b) (10 pts) Find the 3rd Taylor Polynomial for $g(x)$ centered at the point $x_0 = 0$.

$g(0) = \ln(2)$,

$g'(x) = \frac{1}{x+2}$ so $g'(0) = \frac{1}{2}$

$g''(x) = \frac{-1}{(x+2)^2}$ so $g''(0) = -\frac{1}{4}$

$g'''(x) = \frac{2}{(x+2)^3}$ so $g'''(0) = \frac{1}{4}$.

Then $p_3(x) = \ln(2) + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{24}$.

3. Consider the following integrals. In both determine where the integral is "improper", rewrite it with the appropriate limit(s), and determine its convergence. You should specify the number it converges to if it converges, or specify if it approaches ∞, $-\infty$ or DNE.

(a) (10 pts) $\int_1^\infty \frac{dx}{x^2+1}$

This is

$$\lim_{b\to\infty} \int_1^b \frac{dx}{x^2+1} = \lim_{b\to\infty} \arctan(b) - \arctan(1) = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}.$$

(b) (10 pts) $\int_0^1 \frac{dx}{\sqrt[3]{x}}$

This is

$$\lim_{a\to 0^+} \int_a^1 \frac{dx}{\sqrt[3]{x}} = \frac{3}{2}(1 - a^{2/3}) = \frac{3}{2}.$$

(c) (10 pts) $\int_0^1 \frac{dx}{x^2-1}$

This is

$$\lim_{b\to 1^-} \int_0^b \frac{dx}{x^2-1} = \lim_{b\to 1^-} \int_0^b \left(\frac{1}{2(x-1)} - \frac{1}{2(x+1)}\right) dx =$$

$$\lim_{b\to 1^-} \frac{1}{2}(\ln|b-1| - \ln|b+1| - \ln|1| + \ln|-1|) = -\infty$$

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4. Consider the integral

$$\int \frac{4x dx}{3 - 2x - x^2}.$$

(a) (10 pts) Solve the integral by completing the square and doing the appropriate trig-substitution. Simplify as much a possible so that $trig_1(arctrig_2(\alpha)$ does not occur in your expression.

When we complete the square we get

$$\int \frac{4x dx}{4 - (x+1)^2} = \int \frac{x dx}{1 - \left(\frac{x+1}{2}\right)^2}.$$

Then we do the substitution $\frac{x+1}{2} = \sin(u)$ so $dx = 2\cos(u)du$ and $x = 2\sin(u) - 1$. Plugging everything in we get

$$2\int \frac{(2\sin(u) - 1)\cos(u)du}{\cos^2(u)} = \int (4\tan(u) - 2\sec(u))du = -4\ln|\cos(u)| - 2\ln|\sec(u) + \tan(u)| + C$$

$$= -4\ln|\cos(\arcsin((x+1)/2))| - 2\ln|\sec(\arcsin((x+1)/2)) + \tan(\arcsin((x+1)/2))| + C =$$

$$-4\ln|\sqrt{3-2x-x^2}/2| - 2\ln|(2+x+1)/\sqrt{3-2x-x^2}| + C$$

(b) (10 pts) Solve the integral using partial fractions.

We have

$$\frac{4x}{3-2x-x^2} = \frac{4x}{(3+x)(1-x)} = \frac{A}{3+x} + \frac{B}{1-x}$$

Multiplying through we have $4x = A - Ax + 3B + Bx$. So $4 = B - A$ and $0 = A + 3B$. So $B = 1$ and $A = -3$. Then the integral is just $-3\ln|3+x| - \ln|1-x| + D$.

(c) (BONUS - 5 pts) Show using log rules that your expressions in (a) and (b) are equivalent.

$$-4\ln|\sqrt{3-2x-x^2}/2| - 2\ln|(2+x+1)/\sqrt{3-2x-x^2}| + C =$$

$$4\ln(2) - 2\ln|(3+x)(1-x)| - 2\ln|x+3| + \ln|(x+3)(1-x)| + C =$$

$$4\ln(2) - (\ln|x+3| + \ln|1-x|) - 2\ln|x+3| + C = 4\ln(2) - 3\ln|3+x| - \ln|1-x| + C.$$

This is equivalent to (b) where $C + 4\ln(2) = D$.(i.e. the antiderivatives computed differ by the constant $4\ln(2)$.)