Fermat and the Quadrature of the Folium of Descartes

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1. INTRODUCTION. The seventeenth was a century rich in mathematical discoveries and also rich in mathematical discussions and controversies. One of these famous confrontations took place between Fermat and Descartes over the problem of tracing tangents to a curve (see the letter of 18 January 1638 [**23**, vol. 2, pp. 129–130] and the letter of 12 November 1638 [**23**, vol. 2, p. 35]). Descartes did not fully understand Fermat's method for the tracing of tangents, and he believed his own solution (presented in *La Géométrie* [**13**, pp. 95–115]) far better. Many of the controversies of the time took the form of *challenges*: one mathematician, usually in possession of a problem (and a solution to the problem), challenged a colleague or even the whole scientific community to solve the problem. Following this custom, Descartes challenged Fermat to find the tangents to an especially complicated curve that he (Descartes) had invented. That curve has ever since borne his name: *the folium of Descartes*. The name "folium" comes from the leaf shape of the curve's loop in the first quadrant (see Figure 1).

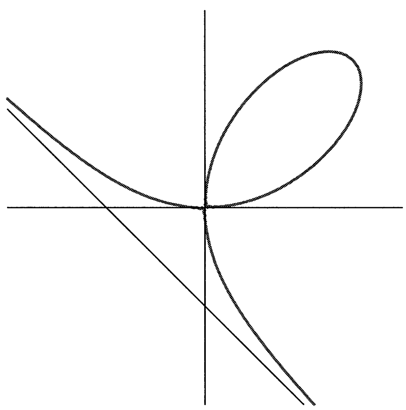

Figure 1. The folium of Descartes, $x^3 + y^3 = 3axy$.

The difficulty that Descartes thought would be impossible for Fermat to overcome was that the proposed curve was a cubic given by an implicit equation: $x^3 + y^3 = 3axy$. Descartes was very proud of his own method for drawing tangents to curves. In his own words: "And I dare say that this is not only the most useful and most general problem in geometry that I know, but even that I have ever desired to know" [**13**, p. 95]. It is not strange then that, when he heard of a method devised by a relatively obscure amateur, Descartes thought that the alleged solution would be just a poor construction that would not pass his test. When Fermat provided the required tangents not only at the vertex of the folium (the only point at which Descartes's method was applicable) but also at *any other point* of the curve, Descartes was obliged to acknowledge the superiority of Fermat's method and his intellectual greatness. See [**8**] to learn more about the problem of tangents and [**19**], [**14**], or [**18**, p. 181] to find out what the controversy was.

What is not nearly so well known, however, is that Fermat solved another problem related to the folium of Descartes: to find the area enclosed by the loop or, in the language of the time, to "square" the loop of the folium. Fermat solved this problem in a paper written in 1658 or 1659 (the exact date is uncertain), in which he collected some results concerning the "quadrature of curves." The paper is known as the *Treatise on Quadrature*, the short form for its puzzling and overwhelmingly long title (see [**15**]).

Today the problem as such is nothing more than a classic exercise that appears in many calculus textbooks, usually in the chapter on the use of polar coordinates in integration. It has even been the object of attention in this MONTHLY, either in the form of a problem (see [**3**], [**4**], [**22**], and [**24**]) or in the form of a short note (see [**17**]). Other problems related to the folium, such as computing its length, the plotting of its graph, or finding its center of gravity have also been tackled (see [**16**], [**5**], [**6**], [**7**], [**1**], [**11**], and [**12**]).

Returning to the folium itself (about which further details can be found at the website [**20**]), we recall that its Cartesian equation is

$$x^3 + y^3 = 3axy,$$

where a is a positive parameter. The loop appears in the first quadrant, (0, 0) is a double point, and each of the two branches of the folium is asymptotic to the line $x + y = -a$, all as shown in Figure 1. (A good reference for the main issues concerning real plane algebraic curves, such as multiple points, asymptotes, behavior near the origin, and so forth, is [**21**, chap. 1].)

Wilson's solution [**24**] to the problem of finding the area enclosed by the loop uses the parametric equations of the folium:

$$\begin{cases} x = \dfrac{3at}{1+t^3}, \\[2ex] y = \dfrac{3at^2}{1+t^3}. \end{cases} \tag{1}$$

The loop is drawn for t varying from 0 to ∞ and the area within is found with the help of a formula, a consequence of Green's theorem, for the area as a curvilinear integral (see [**9**, p. 347]):

$$A = \int_{t=0}^{t=\infty} x\,dy = -\int_{t=0}^{t=\infty} y\,dx = \frac{3}{2}a^2. \tag{2}$$

Bullard [**3**] generalizes the folium and asks for the area enclosed by the loop of the curve with equation

$$x^{2q+1} + y^{2q+1} = (2q+1)ax^q y^q, \tag{3}$$

where q is a positive integer. His problem also asks one to find the area trapped between the generalized folium and its asymptote, the line $x + y = (-1)^q a$.

Bullard's own solution [**4**] changes the folium equation to polar coordinates (r, θ) and uses the formula for the area enclosed by the arc of a polar curve and the radius vectors from the origin to the endpoints of the arc (see [**10**, p. 275]):

$$A = \frac{1}{2}\int_{\theta_1}^{\theta_2} r^2\,d\theta.$$

In the case of the generalized folium (3), the equation in polar coordinates is

$$r = \frac{(2q+1)a\tan^q\theta\sec\theta}{1+\tan^{2q+1}\theta},$$

and the corresponding area within the loop is

$$A(q) = \frac{1}{2}\int_0^{\pi/2} r^2\,d\theta = \frac{2q+1}{2}\,a^2.$$

Remark. The area between the curve and its asymptote is exactly the same as the area of the loop.

Lastly, Johnson [**17**] uses the parametric equations (1) again, but he finds the area of the loop thanks to the symmetric form of the expression for the area given by (2) (see [**10**, p. 273]):

$$A = \frac{1}{2}\int_{t_1}^{t_2}(x\,dy - y\,dx),$$

an expression that, as $t = y/x$ and $dt = (x\,dy - y\,dx)/x^2$ along the curve under consideration, reduces to

$$A = \frac{1}{2}\int_{t_1}^{t_2} x^2\,dt.$$

This makes the calculation of A very easy in the case of the folium of Descartes,

$$A = \frac{9}{2}a^2\int_0^\infty \frac{t^2}{(1+t^3)^2}\,dt = \frac{3}{2}a^2.$$

In the case of Bullard's generalization (3), the parametric equations are

$$\begin{cases} x = \dfrac{(2q+1)at^q}{1+t^{2q+1}}, \\[2ex] y = \dfrac{(2q+1)at^{q+1}}{1+t^{2q+1}}, \end{cases}$$

and the area of the loop is calculated as easily as in the case of the folium:

$$A(q) = \frac{(2q+1)^2}{2}a^2\int_0^\infty \frac{t^{2q}}{(1+t^{2q+1})^2}\,dt = \frac{2q+1}{2}a^2.$$

2. FERMAT'S METHOD OF QUADRATURE. In the first part of the *Treatise on Quadrature*, Fermat obtains formulas to square all the "higher parabolas" with equations $y = x^{m/n}$ for positive integers m and n,

$$\int_0^b x^{m/n}\,dx = \frac{n}{m+n}\,b^{(m+n)/n},$$

and to square all the "higher hyperbolas," whose equations are $y = x^{-m/n}$ with m and n positive integers and $m > n$,

$$\int_b^\infty x^{-m/n}\,dx = \frac{n}{m-n}\,\frac{1}{b^{(m-n)/n}}.$$

(The case of the hyperbola $y = x^{-m/n}$ for $m < n$ can be reduced to the quadrature of the hyperbola $x = y^{-n/m}$ obtained by interchanging the coordinate axes. In this way, the only hyperbola that eludes Fermat's grasp is, obviously, $y = x^{-1}$. See [**2**] for the details.)

In the second part of the *Treatise*, Fermat tackles the quadrature of more complicated curves. Fermat considers only curves drawn in the first quadrant, and the areas he finds are of two types: the area enclosed by a simple curve and its intersections with the coordinate axes (see Figure 2) and the area enclosed by a curve, its asymptote, and a given finite ordinate (see Figure 3). His method is based essentially on a particular case of what is today known as the formula for integration by parts.

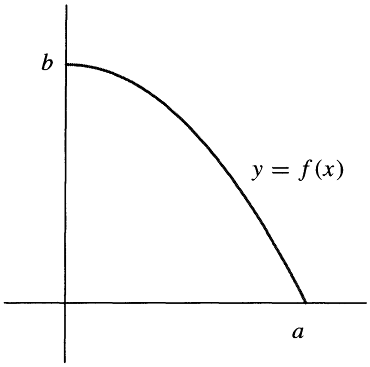

Figure 2.

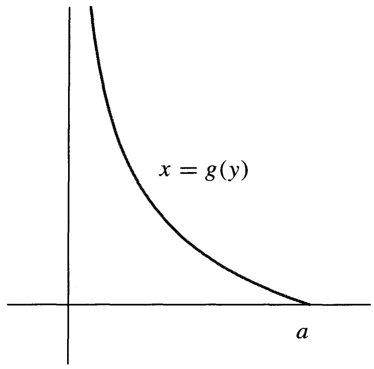

Figure 3.

Theorem 1 (Parts Theorem). *If a positive decreasing function cuts the x-axis on a and the y-axis on b (see Figure 2), then*

$$\int_0^a y^m\,dx = m\int_0^b y^{m-1}x\,dy. \tag{4}$$

Fermat presents this important tool for the squaring of curves without proof. It is very likely, though, that Fermat had learned of the method from Pascal, who had corresponded with him about it (see the letter of 16 February 1659 [**23**, vol. 2, pp. 430 and 436]). Fermat's formulation is geometric but with a good dose of algebra, whereas Pascal's formulation and proof are purely geometric.

The result of the Parts Theorem really is a special case of integration by parts. Referring to Figure 2, we integrate by parts and obtain

$$\int_0^a y^m\,dx = \left[xy^m\right]_0^a - m\int_0^a xy^{m-1}y'\,dx.$$

The term in brackets is 0. Applying the change of variable

$$x = f^{-1}(y), \qquad y'dx = dy, \tag{5}$$

to the integral on the right-hand side, we find that

$$\int_0^a y^m\,dx = -m\int_0^a xy^{m-1}y'\,dx = -m\int_b^0 xy^{m-1}\,dy = m\int_0^b xy^{m-1}\,dy.$$

Fermat had no problem applying the result even when one of the integral limits was ∞. That is to say, the conclusion of the Parts Theorem is valid even in the case where the curve has an asymptote, as shown in Figure 3:

$$\int_0^\infty x^m\,dy = m\int_0^a x^{m-1}y\,dx.$$

We should remark in passing that Fermat did not give priority to one coordinate axis or the other. In each problem he used whichever axis was most convenient for his purposes.

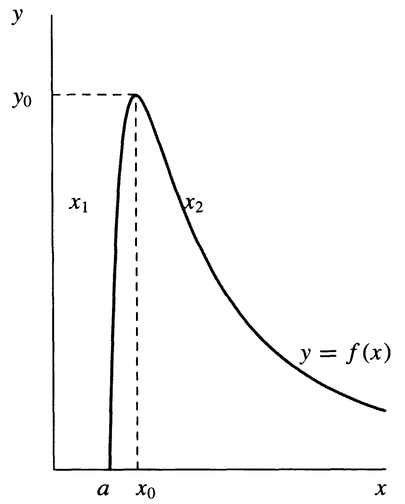

Figure 4.

If a function was not decreasing, Fermat appealed to his *method of maxima and minima* to determine a point such that the graph could be separated into pieces to which the Parts Theorem was directly applicable (see Figure 4). Formula (4), in this case, must be applied separately to each monotone portion of the graph,

$$\int_a^\infty y^m\,dx = \int_a^{x_0} y^m\,dx + \int_{x_0}^\infty y^m\,dx. \tag{6}$$

For the first integral on the right-hand side of (6), we must consider $x_1 = g^{-1}(y)$, where $g = f|[a, x_0]$, in order to apply the change of variable (5):

$$\int_a^{x_0} y^m\,dx = \left[xy^m\right]_a^{x_0} - m\int_a^{x_0} xy^{m-1}y'\,dx = x_0y_0^m - m\int_0^{y_0} y^{m-1}x_1\,dy. \tag{7}$$

As for the second summand in (6), the inverse function to be taken in (5) is $x_2 = h^{-1}(y)$, where $h = f|[x_0, \infty]$:

$$\int_{x_0}^\infty y^m\,dx = \left[xy^m\right]_{x_0}^\infty - m\int_{x_0}^\infty xy^{m-1}y'\,dx = -x_0y_0^m + m\int_0^{y_0} y^{m-1}x_2\,dy. \tag{8}$$

Adding (7) and (8) we finally obtain

$$\int_a^\infty y^m\,dx = m\int_0^{y_0} y^{m-1}(x_2 - x_1)\,dy. \tag{9}$$

In fact, Fermat considered only the case in which the function has two monotone portions, but there is no reason why it should not be applicable to a function with any finite number of monotone pieces. In this case, in order to obtain the equivalent of formula (9), careful attention must be paid to the different pieces and their corresponding limits of integration.

3. THE QUADRATURE OF THE FOLIUM. We are now ready to see how Fermat applied his method. We consider the loop of the folium whose equation is $x^3 + y^3 = 3axy$ (see Figure 5). Notice that in Figure 5, y_0 denotes the y-coordinate of the highest point of the folium. To begin, Fermat made the change of variable

$$x = uy^2, \tag{10}$$

after which the equation of the folium becomes

$$u^3y^6 + y^3 = 3auy^3$$

or, in a different form,

$$y^3 = \frac{3au - 1}{u^3}.$$

The graph of the function $f(u) = (3au - 1)/u^3$ is seen in Figure 6. Fermat asserted that the region enclosed by the loop of the folium had been transformed into the region between the graph of $f(u)$ and its asymptote (the x-axis) over the interval $[1/3a, \infty)$.

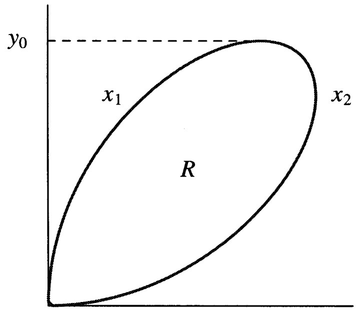

Figure 5. $x^3 + y^3 = 3axy$.

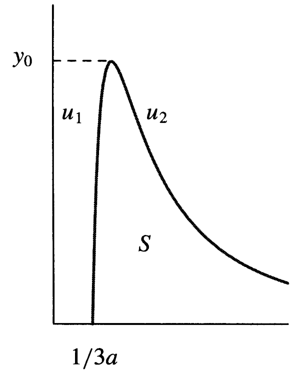

Figure 6. $f(u) = (3au - 1)/u^3$.

The variables x_i and u_i in Figures 5 and 6 are related by

$$x_i = u_iy^2.$$

Denoting by A the area within the loop of the folium, we can write

$$A = \int_0^{y_0} (x_2 - x_1)\, dy = \int_0^{y_0} (u_2 - u_1)y^2\, dy.$$

We now apply the Parts Theorem (9) backwards,

$$\int_0^{y_0} (u_2 - u_1)y^2\, dy = \frac{1}{3}\int_{1/3a}^{\infty} y^3\, du.$$

For Fermat there was no problem in integrating y^3 as a function of the variable u:

$$\int_{1/3a}^{\infty} y^3\, du = \int_{1/3a}^{\infty} \frac{3au-1}{u^3}\, du = \frac{3}{2}a^2.$$

The weak point in Fermat's reasoning was his assertion that the area of the transformed region is the same as the area of the loop of the folium. We can justify this assumption by thinking of Fermat's change of variable $x = uy^2$ as the transformation of plane coordinates

$$(x, y) \mapsto (u, y).$$

The Jacobian of this mapping (i.e., the factor that determines how areas transform) is readily computed:

$$\frac{\partial(x, y)}{\partial(u, y)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial y} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial y} \end{vmatrix} = \begin{vmatrix} y^2 & 2uy \\ 0 & 1 \end{vmatrix} = y^2.$$

The region R of the xy-plane enclosed by the loop of the folium,

$$R = \big\{(x, y) : x_1 \le x \le x_2, 0 < y \le y_0\big\},$$

is transformed into the region S of the uy-plane

$$S = \big\{(u, y) : u_1 \le u \le u_2, 0 < y \le y_0\big\},$$

which can also be described as

$$S = \left\{(u, y) : \frac{1}{3a} \le u \le \infty, 0 < y \le \left(\frac{3au-1}{u^3}\right)^{1/3}\right\}.$$

The area of the loop can then be calculated as follows:

$$A = \iint_R dx\, dy = \int_{1/3a}^{\infty} \int_0^{\left(\frac{3au-1}{u^3}\right)^{1/3}} y^2\, dy\, du$$
$$= \int_{1/3a}^{\infty} \left[\frac{y^3}{3}\right]_0^{\left(\frac{3au-1}{u^3}\right)^{1/3}} du = \frac{1}{3}\int_{1/3a}^{\infty} \frac{3au-1}{u^3}\, du = \frac{3}{2}a^2.$$

4. GENERALIZED FOLIA. Bullard's generalized folium with equation (3), can be treated in exactly the same way. Proceeding *à la Fermat* we make the following change of variable:

$$x = uy^{(q+1)/q}.$$

Fermat himself would never have made such a change of variable involving a fractional exponent. Instead, he might have used $x^q = u^q y^{q+1}$, but nowhere in the *Treatise* is there such a change with $q \ge 2$.

The folium (3) is transformed into the curve with equation

$$y^{(2q+1)/q} = \frac{(2q+1)au^q - 1}{u^{2q+1}},$$

a curve whose graph is similar to the one in Figure 6. The chain of integrals that arose in Fermat's method translates in the generalized setting to

$$\begin{aligned} A(q) &= \int_0^{y_0} (x_2 - x_1)\,dy = \int_0^{y_0} (u_2 - u_1) y^{(q+1)/q}\,dy \\ &\overset{*}{=} \frac{1}{\frac{q+1}{q} + 1} \int_{((2q+1)a)^{-1/q}}^{\infty} y^{\frac{q+1}{q}+1}\,du \\ &= \frac{q}{2q+1} \int_{((2q+1)a)^{-1/q}}^{\infty} \frac{(2q+1)au^q - 1}{u^{2q+1}}\,du = \frac{2q+1}{2}a^2. \end{aligned}$$

The step $(*)$ is the only one that Fermat would not have been prepared to accept. Fermat's (and Pascal's) Parts Theorem was clearly a theorem valid only for *positive integer* exponents.

So much for Bullard's generalized folia. But we can go a little further than Bullard and "generalize" his folia even more. We consider the following one-parameter family of "loops":

$$x^\alpha + y^\alpha = x^{(\alpha-1)/2} y^{(\alpha-1)/2} \qquad (\alpha > 1). \tag{11}$$

We restrict attention to the first quadrant (otherwise x^α and y^α do not make sense), and we take the coefficient of the monomial on the right-hand side to be 1 in order to simplify matters.

The graphs of these equations really are loops contained in the unit square $[0, 1] \times [0, 1]$. To convince oneself of this fact, it is enough to intersect the curve (11) with the straight line $y = kx$ $(k > 0)$ to see that it has exactly two intersection points, namely,

$$(0, 0), \ \left(\frac{k^{(\alpha-1)/2}}{1 + k^\alpha}, \frac{k^{(\alpha+1)/2}}{1 + k^\alpha}\right).$$

As for containment in the unit square, assume that $y \geq x > 0$ (the dual case is similar). Then using (11) we can write

$$y^\alpha < x^\alpha + y^\alpha = x^{(\alpha-1)/2} y^{(\alpha-1)/2} \leq y^{\alpha-1}.$$

As $\alpha > 1$ the fact that $y^\alpha < y^{\alpha-1}$ implies that $y < 1$, and consequently $x < 1$. The loops in this one-parameter family of loops can be considered continuous deformations of the triangle in the first quadrant that is bounded by the line $x + y = 1$ and the coordinate axes (see Figure 7).

Again exploiting Fermat's method, we perform the change of variable

$$x = uy^{(\alpha+1)/(\alpha-1)},$$

which maps the loop (11) to the curve with equation

$$y^{2\alpha/(\alpha-1)} = \frac{u^{(\alpha-1)/2} - 1}{u^\alpha},$$

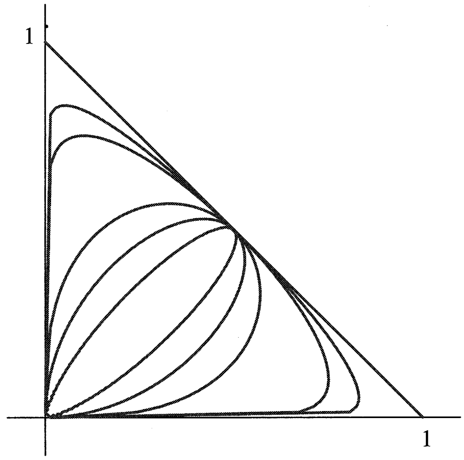

Figure 7. $x^\alpha + y^\alpha = x^{\frac{\alpha-1}{2}} y^{\frac{\alpha-1}{2}}$ $(\alpha > 1)$.

whose graph is once more similar to the one pictured in Figure 6. The chain of integrals that leads to the area of the loop is, as before,

$$A(\alpha) = \int_0^{y_0} (x_2 - x_1)\, dy = \int_0^{y_0} (u_2 - u_1) y^{(\alpha+1)/(\alpha-1)}\, dy$$
$$= \frac{1}{\frac{\alpha+1}{\alpha-1} + 1} \int_1^\infty y^{\frac{\alpha+1}{\alpha-1}+1}\, du = \frac{\alpha-1}{2\alpha} \int_1^\infty \frac{u^{(\alpha-1)/2} - 1}{u^\alpha}\, du = \frac{1}{2\alpha}.$$

5. A NEW FAMILY OF FOLIA. Bullard's generalization of the folium of Descartes is a very natural one for two reasons. In the first place, the shape of the generalized folium is very similar to the original. Second, and more important, is the fact that the equation of the generalized folium follows the pattern

$$P_{2q+1}(x, y) = P_{2q}(x, y),$$

where P_n denotes a homogeneous polynomial of degree n. This pattern allows for the parametrization of the curve through the parameter $t = y/x$. As a result, either the change to polar coordinates or the method used in Johnson's note **[17]** is applicable.

There is, however, another natural family of curves that generalize the folium of Descartes—namely, the curves defined by equations of the type

$$x^{2q+1} + y^{2q+1} = axy,$$

where $a > 0$ and q is a positive integer, whose graphs are depicted in Figures 8 and 9. (Figure 8 exhibits the global character of such a curve, while Figure 9 plots the folia corresponding to various values of q.) We consider only the case $a = 1$ for the sake of simplicity.

For $q > 1$, the asymptote of the curve is the line $x + y = 0$ and, as q increases, the loop seems to approach the boundary of the unit square in the first quadrant, as Figure 9 suggests.

Following Fermat's procedure, we make the change of variable

$$x = uy^{2q}.$$

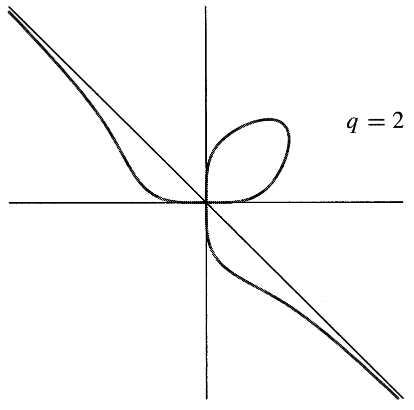

Figure 8. $x^{2q+1} + y^{2q+1} = xy$.

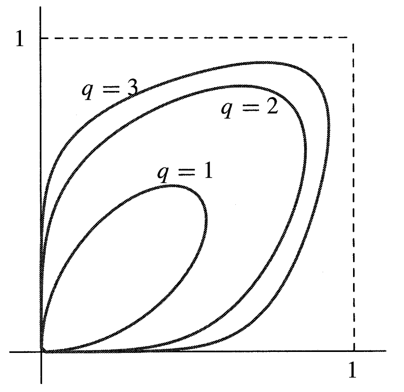

Figure 9. $x^{2q+1} + y^{2q+1} = xy$.

After simplification, the equation of the transformed curve becomes

$$u^{2q+1} y^{(2q+1)(2q-1)} + 1 = u,$$

or, thinking of Fermat's chain of integrals,

$$y^{2q+1} = \left(\frac{u-1}{u^{2q+1}} \right)^{1/(2q-1)}.$$

The graph of a function

$$f(u) = \left(\frac{u-1}{u^{2q+1}} \right)^{1/(2q-1)}$$

for u between 1 and ∞ is similar to the one shown in Figure 6 (which is precisely the case $q = 1$, the folium of Descartes). As earlier, the region within the loop is transformed into the region between the graph of f and the asymptote $y = 0$ above the interval $[1, \infty)$.

As before, we apply (9) to each of the intervals on which f is monotone and make in the two integrals so obtained the change of variable

$$x_i = u_i y^{2q}.$$

If $\tilde{A}(q)$ denotes the area within the loop, we obtain

$$\tilde{A}(q) = \int_0^{y_0} (x_2 - x_1)\, dy = \int_0^{y_0} (u_2 - u_1) y^{2q}\, dy = \frac{1}{2q+1} \int_1^{\infty} y^{2q+1}\, du.$$

But

$$y^{2q+1} = \left(\frac{u-1}{u^{2q+1}} \right)^{1/(2q-1)},$$

so

$$\tilde{A}(q) = \frac{1}{2q+1} \int_1^{\infty} \left(\frac{u-1}{u^{2q+1}} \right)^{1/(2q-1)} du.$$

This last integral is not trivial and requires some work. First of all, the change of variables $u - 1 = t$ transforms it into

$$\tilde{A}(q) = \frac{1}{2q+1} \int_0^{\infty} \frac{t^{1/(2q-1)}}{(1+t)^{(2q+1)/(2q-1)}}\, dt, \tag{12}$$

which is related to Euler's beta function. We remind the reader of the latter function's definition (see [**9**, pp. 335–339]). For real x and y,

$$\mathrm{B}(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}\, dt. \tag{13}$$

If either x or y is less than 1, the integral is improper but convergent. It is also common to present (13) as follows: for real numbers m and n,

$$\mathrm{B}(m+1, n+1) = \int_0^1 x^m (1-x)^n\, dx.$$

With the change of variable $x = t/(1+t)$ we can rewrite this in the form

$$\mathrm{B}(m+1, n+1) = \int_0^{\infty} \frac{t^m}{(1+t)^{m+n+2}}\, dx.$$

Motivated by this last representation of the beta function, in (12) we set

$$\frac{1}{2q-1} = m,$$

which gives

$$\frac{2q+1}{2q-1} = 2m + 1 = m + (m-1) + 2.$$

Equation (12) can thus be restated as

$$\tilde{A}(q) = \frac{1}{2q+1} \int_0^{\infty} \frac{t^m}{(1+t)^{m+(m-1)+2}}\, dt = \frac{1}{2q+1} \mathrm{B}(m+1, m). \tag{14}$$

We recall two properties of the beta function:

(i) $\mathrm{B}(p, q) = \mathrm{B}(q, p)$;

(ii) $\mathrm{B}(p, q+1) = \frac{p}{p+q} \mathrm{B}(p, q)$.

Using (i) and (ii) in (14), we arrive at

$$\tilde{A}(q) = \frac{1}{2q+1} \mathrm{B}(m+1, m) = \frac{1}{2q+1} \mathrm{B}(m, m+1) = \frac{1}{2q+1} \cdot \frac{1}{2} \mathrm{B}(m, m).$$

Finally, we obtain a compact expression for $\tilde{A}(q)$:

$$\tilde{A}(q) = \frac{1}{2(2q+1)} \cdot \mathrm{B}\left(\frac{1}{2q-1}, \frac{1}{2q-1}\right).$$

As a particular case, for $q = 1$ we obtain the area of the loop of a folium of Descartes (for the parameter $a = 1/3$):

$$\tilde{A}(1) = \frac{1}{6}\,\mathrm{B}(1, 1) = \frac{1}{6}.$$

It is also easy to see that, as we had suspected for graphical reasons, for large q the loop tends to exhaust the unit square:

$$\lim_{q\to\infty} \tilde{A}(q) = \lim_{q\to\infty} \frac{1}{2(2q+1)} \cdot \mathrm{B}\left(\frac{1}{2q-1}, \frac{1}{2q-1}\right) = 1.$$

Further generalization of this new family of folia would lead us to consider curves with equations

$$x^\alpha + y^\alpha = x^\beta y^\gamma \qquad (\alpha > 2;\;\; \beta, \gamma > 0;\;\; \beta + \gamma < \alpha),$$

which can be shown to include loops contained in the unit square $[0, 1] \times [0, 1]$ similar to the ones considered hitherto. With the same type of arguments and self-explanatory notation we would obtain

$$\bar{A}(\alpha, \beta, \gamma) = \frac{1}{2\alpha} \cdot \mathrm{B}\left(\frac{\alpha - \beta + \gamma}{\alpha(\alpha - \beta - \gamma)}, \frac{\alpha + \beta - \gamma}{\alpha(\alpha - \beta - \gamma)}\right).$$

6. CONCLUDING REMARKS. It is really surprising that, with the help of very elementary geometric and algebraic notions, Fermat managed to develop such an original technique for the quadrature of curves. We must admit that, even with our powerful algebraic notation and the still more powerful identification *area* = *integral*, the method we have presented is already quite original. Imagine then, how difficult it would have been for Fermat, who spoke in a predominantly geometric language only about *squaring* x and *squaring* y! We cannot but stand in awe and admiration of his genius.

It is worthwhile mentioning that the *Treatise on Quadrature* [**15**] is a quite obscure paper that is difficult to read. The great Huygens remarked, in reference to this paper: "[T]his treatise has been published with many mistakes and it is so obscure (with proofs redolent of error) that I have been unable to make any sense of it" (letter of Huygens to Leibniz, 1 September 1691, [**23**, vol. 4, p. 137]). Since then, the more obscure parts of the *Treatise* have been largely ignored by historians of mathematics: only two authors seem to have glimpsed the possibilities of the method, Zeuthen [**25**] and Mahoney [**18**], though both were far from exhausting its possibilities.

The method of Fermat, of which we have only seen a small part, deserves further study. In some instances, as is the case of the generalized folia (11), the area enclosed by the loop is not easy to obtain, either by changing to polar coordinates or by any traditional method of the calculus. In these cases, Fermat's simple change of variable and his clever use of the Parts Theorem offer an unexpected way to solve them.

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REFERENCES

1. M. C. Baudin, Discussions: Relating to the folium of Descartes, this MONTHLY **23** (1916) 90–92.
2. C. B. Boyer, Fermat's integration of X^n, *Nat. Math. Mag.* **20** (1945) 29–32.
3. J. A. Bullard, Problem 410, this MONTHLY **23** (1916) 210.
4. ———, Solution of problem 410, this MONTHLY **24** (1917) 86.
5. G. W. Coakley, Solutions of problems 159 and 160 in Solutions of Exercises, *Annals of Math.* **4** (1888) 58–60.
6. J. M. Colaw and G. B. M. Zerr, Problem 119, this MONTHLY **8** (1901) 22.
7. ———, Solution of problem 119, this MONTHLY **8** (1901) 260.
8. J. L. Coolidge, The story of tangents, this MONTHLY **58** (1951) 449–462.
9. R. Courant, *Differential and Integral Calculus*, vol. 2, Blackie & Son, Glasgow, 1936.
10. ———, *Differential and Integral Calculus*, vol. 1, 2nd ed., Blackie & Son, Glasgow, 1937.
11. A. Danzl, Problem E1009, this MONTHLY **59** (1952) 180–181.
12. ———, Solution of problem E1009, this MONTHLY **59** (1952) 697–698.
13. R. Descartes, *The Geometry of René Descartes. (With a facsimile of the first edition, 1637)* (trans. D. E. Smith and M. L. Latham), Dover, New York, 1954.
14. J. M. C. Duhamel, Mémoire sur la méthode de maxima et minima de Fermat et sur les méthodes des tangentes de Fermat et Descartes, *Mémoires de l'Académie des Sciences de l'Institut Impérial de France* **32** (1864) 269–330.
15. P. de Fermat, Sur la transformation et la simplification des équations de lieux, pour la comparaison sous toutes les formes des aires curvilignes, soit entre elles, soit avec les rectilignes, et en même temps, sur l'emploi de la progréssion géométrique pour la quadrature des paraboles et hyperboles a l'infini, in [**23**, vol. 3], pp. 216–237; first published in 1679.
16. R. H. Graves, Problems 159 and 160 in Exercises, *Annals of Math.* **3** (1887) 190.
17. R. A. Johnson, A note on areas, this MONTHLY **52** (1945) 209–210.
18. M. S. Mahoney, *The Mathematical Career of Pierre de Fermat, 1601–1665*, Princeton University Press, Princeton, 1973.
19. Turnbull WWW Server, http://turnbull.mcs.st-and.ac.uk/history/ Mathematicians/Fermat.html.
20. ———, http://turnbull.mcs.st-and.ac.uk/history/Curves/ Foliumd.html.
21. E. J. F. Primrose, *Plane Algebraic Curves*, Macmillan, London, 1955.
22. C. N. Schmall, Problem 379, this MONTHLY **22** (1915) 98.
23. P. Tannery and C. Henry, eds. *Œuvres de Pierre Fermat*, 4 vols. plus suppl., Gauthier–Villars, Paris, 1894–1912.
24. E. B. Wilson, Solution of problem 379, this MONTHLY **22** (1915) 343.
25. H. G. Zeuthen, Notes sur l'histoire des matémathiques, (suite) IV: Sur les quadratures avant le calcul intégral, et en particulier sur celles de Fermat, *Oversigt over det Kongelige Danske Videnskabernes Selskabs. Forhandlinger* (1895) 37–80, part 4 of a paper started in 1893 in the same Bulletin.

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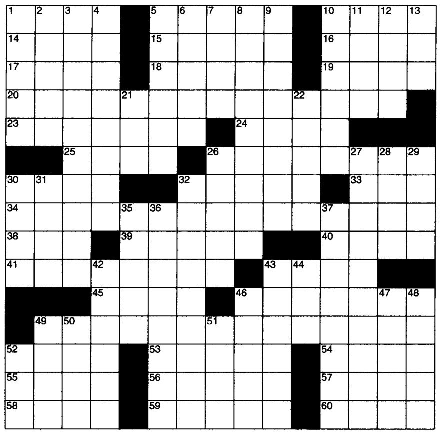
Ill Logical



ACROSS

1 ________ lock key
5 Hold fast to the truth as ______ (Buddha)
10 Not 27 down
14 Concerning; or, broken part of a bridle?
15 Daughter of Aeetes and wife of Jason
16 Not odd
17 Not now and not much later
18 Nor' ________
19 Pistol or malt liquor
20 Conjunctive battle at the Little Bighorn?
23 2.54 centimeters or 1/198 rod
24 Scheme or outline
25 Re or se follower
26 Fabric, not 51 down
30 Avoid or eschew
32 Cactus-like lilies or myrrh's partner
33 Org. for teachers and staff
34 Alexander the Great's negative stroke?
38 Word with old or iron
39 O.K. gunfighters Morgan, Virgil, and Wyatt
40 Meaning for 35 down
41 Equivalent to Lake Constance
43 Den of fox or lion
45 "Very interesting" actor Johnson
46 Coke or Pepsi
49 Oliver Twist's disjunctive supplication?
52 Not warp
53 Double-reed woodwinds
54 Type of dropper or plate
55 South African archbishop, or ballet skirt
56 Branch of peace or salad oil
57 *Oxalis crenata* and *tuberosa*; or, Maple and Mathematica in the voc. case?
58 October birthstone, not tourmaline
59 Not insertions
60 Not ones and not hundreds

DOWN

1 Networking company; or, defective shortening?
2 Singular subject or object
3 William Jennings Bryan did it to Scopes and Marcia Clark did it to O.J.
4 Descartes was, so he was
5 Fine or mulct
6 Restraining strap or three of a kind
7 Broadband from AOL or SBC
8 Do not mix these
9 Having a petroselinum garnish, as potatoes or carrots
10 Disavow or unsay
11 Shakespeare's river or cosmetics company
12 Neuter; or, German money
13 Tolkien tree creature or otolaryngologist
21 Not the beginning and not the middle
22 Ankles and the bones thereof
26 A vertical line does not have one
27 Goal of yoga or Zen
28 Type of Dodge or sign
29 Ship channel and gangster's pistol
30 Strike breaker or skin disease
31 French novelist or science fiction award
32 Not uncongenial
35 Matisse or Poincaré
36 Spaghetti western actor Clint
37 ________ of the world (John 17:14)
42 Scolding, or surprising news
43 Not gains
44 Hustle and bustle
46 Jobs or Wozniak
47 James Bond or one of the same rank?
48 Muscle-building exercise or politician's bane
49 Hot and sour or egg drop
50 Small quantity or Greek letter
51 This and 26 across are fabrics
52 Controversial trade assn.; or, binary permutation?

Contributed by Harold Boas, Texas A & M University