ed. by J. Fauvel & J. Gray

Newton's letter to Leibniz

12.C2 From the Epistola Posterior (1676)

(a) I can hardly tell with what pleasure I have read the letters of those very distinguished men Leibniz and Tschirnhaus. Leibniz's method for obtaining convergent series is certainly very elegant, and it would have sufficiently revealed the genius of its author, even if he had written nothing else. But what he has scattered elsewhere throughout his letter is most worthy of his reputation—it leads us also to hope for very great things from him. The variety of ways by which the same goal is approached has given me the greater pleasure, because three methods of arriving at series of that kind had already become known to me, so that I could scarcely expect a new one to be communicated to us. One of mine I have described before; I now add another, namely, that by which I first chanced on these series—for I chanced on them before I knew the divisions and extractions of roots which I now use. And an explanation of this will serve to lay bare, what Leibniz desires from me, the basis of the theorem set forth near the beginning of the former letter.

At the beginning of my mathematical studies, when I had met with the works of our celebrated Wallis, on considering the series by the intercalation of which he himself exhibits the area of the circle and the hyperbola, the fact that, in the series of curves whose common base or axis is x and the ordinates

$$(1-x^2)^{\frac{0}{2}},\quad (1-x^2)^{\frac{1}{2}},\quad (1-x^2)^{\frac{2}{2}},\quad (1-x^2)^{\frac{3}{2}},\quad (1-x^2)^{\frac{4}{2}},\quad (1-x^2)^{\frac{5}{2}},\quad \text{etc.},$$

if the areas of every other of them, namely

$$x,\quad x-\tfrac{1}{3}x^3,\quad x-\tfrac{2}{3}x^3+\tfrac{1}{5}x^5,\quad x-\tfrac{3}{3}x^3+\tfrac{3}{5}x^5-\tfrac{1}{7}x^7,\quad \text{etc.}$$

could be interpolated, we should have the areas of the intermediate ones, of which the first $(1-x^2)^{\frac{1}{2}}$ is the circle: in order to interpolate these series I noted that in all of them the first term was x and the second terms $\frac{0}{3}x^3$, $\frac{1}{3}x^3$, $\frac{2}{3}x^3$, $\frac{3}{3}x^3$, etc., were in arithmetical progression, and hence that the first two terms of the series to be intercalated ought to be $x-\frac{1}{3}(\frac{1}{2}x^3)$, $x-\frac{1}{3}(\frac{3}{2}x^3)$, $x-\frac{1}{3}(\frac{5}{2}x^3)$, etc. To intercalate the rest I began to reflect that the denominators $1, 3, 5, 7$, etc. were in arithmetical progression, so that the numerical coefficients of the numerators only were still in need of investigation. But in the alternately given areas these were the figures of powers of the number 11, namely of these, $11^0, 11^1, 11^2, 11^3, 11^4$, that is, first 1; then 1, 1; thirdly, 1, 2, 1; fourthly 1, 3, 3, 1; fifthly 1, 4, 6, 4, 1, etc. And so I began to inquire how the remaining figures in these series could be derived from the first two given figures, and I found that on putting m for the second figure, the rest would be produced by continual multiplication of the terms of this series,

$$\frac{m-0}{1}\times\frac{m-1}{2}\times\frac{m-2}{3}\times\frac{m-3}{4}\times\frac{m-4}{5},\quad \text{etc.}$$

For example, let $m = 4$, and $4\times\frac{1}{2}(m-1)$, that is 6 will be the third term, and $6\times\frac{1}{3}(m-2)$, that is 4 the fourth, and $4\times\frac{1}{4}(m-3)$, that is 1 the fifth, and $1\times\frac{1}{5}(m-4)$, that is 0 the sixth, at which term in this case the series stops. Accordingly, I applied this rule for interposing series among series, and since, for the circle, the second term was $\frac{1}{3}(\frac{1}{2}x^3)$, I put $m=\frac{1}{2}$, and the terms arising were

$$\frac{1}{2}\times\frac{\frac{1}{2}-1}{2}\ \text{or}\ -\tfrac{1}{8},\qquad -\frac{1}{8}\times\frac{\frac{1}{2}-2}{3}\ \text{or}\ +\tfrac{1}{16},\qquad \frac{1}{16}\times\frac{\frac{1}{2}-3}{4}\ \text{or}\ -\tfrac{5}{128},$$

and so to infinity. Whence I came to understand that the area of the circular segment which I wanted was

$$x - \frac{\frac{1}{2}x^3}{3} - \frac{\frac{1}{8}x^5}{5} - \frac{\frac{1}{16}x^7}{7} - \frac{\frac{5}{128}x^9}{9} \quad \text{etc.}$$

And by the same reasoning the areas of the remaining curves, which were to be inserted, were likewise obtained: as also the area of the hyperbola and of the other alternate curves in this series $(1 + x^2)^{\frac{0}{2}}$, $(1 + x^2)^{\frac{1}{2}}$, $(1 + x^2)^{\frac{2}{2}}$, $(1 + x^2)^{\frac{3}{2}}$, etc. And the same theory serves to intercalate other series, and that through intervals of two or more terms when they are absent at the same time. This was my first entry upon these studies, and it had certainly escaped my memory, had I not a few weeks ago cast my eye back on some notes.

But when I had learnt this, I immediately began to consider that the terms

$$(1 - x^2)^{\frac{0}{2}}, \quad (1 - x^2)^{\frac{2}{2}}, \quad (1 - x^2)^{\frac{4}{2}}, \quad (1 - x^2)^{\frac{6}{2}}, \quad \text{etc.,}$$

that is to say,

$$1, \quad 1 - x^2, \quad 1 - 2x^2 + x^4, \quad 1 - 3x^2 + 3x^4 - x^6, \quad \text{etc.}$$

could be interpolated in the same way as the areas generated by them: and that nothing else was required for this purpose but to omit the denominators 1, 3, 5, 7, etc., which are in the terms expressing the areas; this means that the coefficients of the terms of the quantity to be intercalated $(1 - x^2)^{\frac{1}{2}}$, or $(1 - x^2)^{\frac{3}{2}}$, or in general $(1 - x^2)^m$, arise by the continued multiplication of the terms of this series

$$m \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4}, \quad \text{etc.,}$$

so that (for example)

$$(1 - x^2)^{\frac{1}{2}} \quad \text{was the value of} \quad 1 - \tfrac{1}{2}x^2 - \tfrac{1}{8}x^4 - \tfrac{1}{16}x^6 \quad \text{etc.,}$$

$$(1 - x^2)^{\frac{3}{2}} \quad \text{of} \quad 1 - \tfrac{3}{2}x^2 + \tfrac{3}{8}x^4 + \tfrac{1}{16}x^6, \quad \text{etc.,}$$

and

$$(1 - x^2)^{\frac{1}{3}} \quad \text{of} \quad 1 - \tfrac{1}{3}x^2 - \tfrac{1}{9}x^4 - \tfrac{5}{81}x^6, \quad \text{etc.}$$

So then the general reduction of radicals into infinite series by that rule, which I laid down at the beginning of my earlier letter became known to me, and that before I was acquainted with the extraction of roots. But once this was known, that other could not long remain hidden from me. For in order to test these processes, I multiplied

$$1 - \tfrac{1}{2}x^2 - \tfrac{1}{8}x^4 - \tfrac{1}{16}x^6, \quad \text{etc.}$$

into itself; and it became $1 - x^2$, the remaining terms vanishing by the continuation of the series to infinity. And even so $1 - \frac{1}{3}x^2 - \frac{1}{9}x^4 - \frac{5}{81}x^6$, etc. multiplied twice into itself also produced $1 - x^2$. And as this was not only sure proof of these conclusions so too it guided me to try whether, conversely, these series, which it thus affirmed to be roots of the quantity $1 - x^2$, might not be extracted out of it in an arithmetical manner. And the matter turned out well. This was the form of the working in square roots.

$$
\begin{array}{l}
1 - x^2 \;(\; 1 - \tfrac{1}{2}x^2 - \tfrac{1}{8}x^4 - \tfrac{1}{16}x^6, \quad \text{etc.} \\
1 \\
\hline
0 - x^2 \\
\quad - x^2 + \tfrac{1}{4}x^4 \\
\hline
\qquad - \tfrac{1}{4}x^4 \\
\qquad - \tfrac{1}{4}x^4 + \tfrac{1}{8}x^6 + \tfrac{1}{64}x^8 \\
\hline
\qquad 0 \quad - \tfrac{1}{8}x^6 - \tfrac{1}{64}x^8.
\end{array}
$$

After getting this clear I have quite given up the interpolation of series, and have made use of these operations only, as giving more natural foundations. Nor was there any secret about reduction by division, an easier affair in any case.

(b) But in that treatise infinite series played no great part. Not a few other things I brought together, among them the method of drawing tangents which the very skilful Sluse communicated to you two or three years ago, about which you wrote back [to him] (on the suggestion of Collins) that the same method had been known to me also. We happened on it by different reasoning: for, as I work it, the matter needs no proof. Nobody, if he possessed my basis, could draw tangents any other way, unless he were deliberately wandering from the straight path. Indeed we do not here stick at equations in radicals involving one or each indefinite quantity, however complicated they may be; but without any reduction of such equations (which would generally render the work endless) the tangent is drawn directly. And the same is true in questions of maxima and minima, and in some others too, of which I am not now speaking. The foundation of these operations is evident enough, in fact; but because I cannot proceed with the explanation of it now, I have preferred to conceal it thus: *6accdæ13eff7i3l9n4o4qrr4s8t12vx*. On this foundation I have also tried to simplify the theories which concern the squaring of curves, and I have arrived at certain general Theorems. And, to be frank, here is the first Theorem.

For any curve let $dz^{\theta} \times (e + fz^{\eta})^{\lambda}$ be the ordinate, standing normal at the end z of the abscissa or the base, where the letters d, e, f denote any given quantities, and θ, η, λ are the indices of the powers of the quantities to which they are attached. Put

$$\frac{\theta + 1}{\eta} = r, \quad \lambda + r = s, \quad \frac{d}{\eta f} \times (e + fz^{\eta})^{\lambda + 1} = Q \quad \text{and} \quad r\eta - \eta = \pi,$$

then the area of the curve will be

$$Q \times \left\{ \frac{z^{\pi}}{s} - \frac{r-1}{s-1} \times \frac{eA}{fz^{\eta}} + \frac{r-2}{s-2} \times \frac{eB}{fz^{\eta}} - \frac{r-3}{s-3} \times \frac{eC}{fz^{\eta}} + \frac{r-4}{s-4} \times \frac{eD}{fz^{\eta}}, \text{ etc.} \right\}$$

the letters A, B, C, D, etc., denoting the terms immediately preceding; that is, A the term z^{π}/s, B the term $-((r-1)/(s-1)) \times (eA)/(fz^{\eta})$, etc. This series, when r is a fraction or a negative number, is continued to infinity; but when r is positive and integral it is continued only to as many terms as there are units in r itself; and so it exhibits the geometrical squaring of the curve. I illustrate the fact by examples.

(c) When I said that almost all problems are soluble I wished to be understood to refer specially to those about which mathematicians have hitherto concerned themselves, or at least those in which mathematical arguments can gain some place. For of course one may imagine others so involved in complicated conditions that we do not succeed in understanding them well enough, and much less in bearing the burden of such long calculations as they require. Nevertheless—lest I seem to have said too much—inverse problems of tangents are within our power, and others more difficult than those, and to solve them I have used a twofold method of which one part is neater, the other more general. At present I have thought fit to register them both by transposed letters, lest, through others obtaining the same result, I should be compelled to change the plan in some respects.

5accdae10effh11i4l3m9n6oqqr8s11t9y3x:11ab3cdd10eæg10ill4m7n6o3p3q6r5s11t8vx,

3acæ4egh5i4l4m5n8oq4r3s6t4v, aaddæcecceiijmmnnooprrrsssssttuu.

This inverse problem of tangents, when the tangent between the point of contact and the axis of the figure is of given length, does not demand these methods. Yet it is that mechanical curve the determination of which depends on the area of an hyperbola. The problem is also of the same kind, when the part of the axis between the tangent and the ordinate is given in length. But I should scarcely have reckoned these cases among the sports of nature. For when in the right-angled triangle, which is formed by that part of the axis, the tangent and the ordinate, the relation of any two sides is defined by any equation, the problem can be solved apart from my general method. But when a part of the axis ending at some point given in position enters the bracket, then the question is apt to work out differently.

The communication of the solution of affected equations by the method of Leibniz will be very agreeable; so too an explanation how he comports himself when the indices of the powers are fractional, as in this equation $20 + x^{\frac{3}{7}} - x^{\frac{6}{5}}y^{\frac{2}{3}} - y^{\frac{7}{11}} = 0$, or surds, as in $(x^{\sqrt{2}} + x^{\sqrt{7}})^{\sqrt[3]{\frac{2}{3}}} = y$, where $\sqrt{2}$ and $\sqrt{7}$ do not mean coefficients of x, but indices of powers or dignities of it, and $\sqrt[3]{\frac{2}{3}}$ means the power of the binomial $x^{\sqrt{2}} + x^{\sqrt{7}}$. The point, I think, is clear by my method, otherwise I should have described it. But a term must at last be set to this wordy letter. The letter of the most excellent Leibniz fully deserved of course that I should give it this more extended reply. And this time I wanted to write in greater detail because I did not believe that your more engaging pursuits should often be interrupted by me with this rather austere kind of writing.

The second anagram runs as follows:

'Una Methodus consistit in extractione fluentis quantitatis ex æquatione simul involvente fluxionem ejus: altera tantum in assumptione Seriei pro quantitate qualibet incognita ex qua cætera commode derivari possunt, & in collatione terminorum homologorum æquationis resultantis, ad eruendos terminos assumptæ seriei.' ('One method consists in extracting a fluent quantity from an equation at the same time involving its fluxion; but another by assuming a series for any unknown quantity whatever, from which the rest could conveniently be derived, and in collecting homologous terms of the resulting equation in order to elicit the terms of the assumed series.')