Exercise 16 *(trapezoidal method for advection)*

Consider the method

$$U_j^{n+1} = U_j^n - \frac{ak}{2h}(U_j^n - U_{j-1}^n + U_j^{n+1} - U_{j-1}^{n+1}). \tag{Ex16a}$$

for the advection equation $u_t + au_x = 0$ on $0 \le x \le 1$ with periodic boundary conditions.

(a) This method can be viewed as the trapezoidal method applied to an ODE system $U'(t) = AU(t)$ arising from a method of lines discretization of the advection equation. What is the matrix A? Don't forget the boundary conditions.

(b) Suppose we want to fix the Courant number ak/h as $k,\ h \to 0$. For what range of Courant numbers will the method be stable if $a > 0$? If $a < 0$? Justify your answers in terms of eigenvalues of the matrix A from part (a) and the stability regions of the trapezoidal method.

(c) Apply von Neumann stability analysis to the method (Ex16a). What is the amplification factor $g(\xi)$?

(d) For what range of ak/h will the CFL condition be satisfied for this method (with periodic boundary conditions)?

(e) Suppose we use the same method (Ex16a) for the initial-boundary value problem with $u(0,t) = g_0(t)$ specified. Since the method has a one-sided stencil, no numerical boundary condition is needed at the right boundary (the formula (Ex16a) can be applied at x_{m+1}). For what range of ak/h will the CFL condition be satisfied in this case? What are the eigenvalues of the A matrix for this case and when will the method be stable?