Math 7670 – Suggested Problems # 4 – Spring 2014

1. Prove Hilbert's syzygy theorem, in the following form: Let $S = k[x_0, \dots, x_n]$, and suppose that M is a finitely generated graded S-module. The length of a minimal free resolution is well defined, and is called $pd_S(M)$. Show that $pd_S(M) \leq n+1$.

   You may use the properties of Tor and the Koszul complex which were stated and/or proved in class.

2. Suppose that S and M are as in the first problem, and also suppose that you are given a graded minimal free resolution of M:

$$0 \longrightarrow F_r \longrightarrow F_{r-1} \longrightarrow \dots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0,$$

   where for each $0 \leq i \leq r$,

$$F_i = \bigoplus_d S(-d)^{b_{id}}.$$

   Define $t_i(M) = \max\{d : b_{id} \neq 0\}$, for $0 \leq i \leq r$.

   (a) We know that there exists an integer m such that for all $\ell \geq m$,

$$Hom((x_0, \dots, x_n)^\ell, M)_{\geq 0} \cong H^0_*(\tilde{M})_{\geq 0}.$$

   Find a possible m, in terms of (possibly only some of) the $t_i(M)$. For simplicity, assume that the length of the resolution r is n (not $n+1$). What power is required if the length of the resolution if $< n$?

   (*Hint:* Consider the exact sequence

$$0 \longrightarrow (x_0, \dots, x_n)^{\ell+1} \longrightarrow (x_0, \dots, x_n)^\ell \longrightarrow k^N(-\ell) \longrightarrow 0,$$

   for some integer N).

   (b) We know that there exists an integer m' such that for all $\ell \geq m'$,

$$Hom((x_0^\ell, \dots, x_n^\ell), M)_{\geq 0} \cong H^0_*(\tilde{M})_{\geq 0}.$$

   Can you find m' (again, in terms of the degrees in the resolution).

   (c) Investigate, for monomial curve ideals in $\mathbf{P}^3$, what powers m and m' give the correct answer, in particular, how optimal are your answers from the first 2 parts of this problem? See the Macaulay2 handout for an example of doing one of these. I suggest that you do examples while attempting to solve parts (a) and (b).

3. Let C be a curve of genus 4 and degree 6 in $\mathbf{P^3}$. Riemann-Roch implies that

$$\chi(\mathcal{O}_C(d) = 6d + 1 - 4 = 6d - 3.$$

   You are told that $H^1(\mathcal{O}_C(d)) = 0$, for $d \geq 2$ (this will also follow from Riemann-Roch). Show that X lies on a quadric surface. Then show that the ideal of C has one quadric and one cubic as minimal generators. In fact, C is a complete intersection of these two polynomials.

4. A coherent sheaf F on $\mathbf{P^n}$ is called m-regular if $H^i(F(m-i)) = 0$, for all $i \geq 1$. Show that if F is m-regular, then F is also $(m+1)$-regular.