ACM 11: Homework #3

Due on Thursday, May 29, 2014,11:59:59 pm

Submit your code in a notebook format and send to acm11spring2014@gmail.com. The name of the nb file is in the format LastFirstHomework3.nb. Follow the homework guidelines in the syllabus

Problem 1

Mathematica as a calculator

Answer the following questions using one line of Mathematica code for each question (if you need to save a number to a variable, or clear the workspace that would not count as a line of code)

a) Is 1304969544928657 a Fibbonachi number? (Hint: use the tests from the Wikipedia page)

b) Is 2305843008139952128 a perfect number?

c) How many prime numbers are less than a billion?

d) Evaluate the 16-th zero of the Riemann zeta function with precision 450.

e) Evaluate

$$\sum_{k=1}^{n} k \sin k + k^2 \cos k$$

and output it using **TraditionalForm**

Problem 2

Volumes of n-dimensional balls and Monte-Carlo method

The volume of an n-dimensional unit ball is given by a formula

$$V_n(R) = \frac{\pi^{\frac{n}{2}} R^n}{\Gamma(\frac{n}{2} + 1)}$$

and the surface area of an n-dimensional sphere is given by

$$S_n(R) = \frac{(n+1)\pi^{\frac{n+1}{2}} R^n}{\Gamma(\frac{n+1}{2} + 1)}$$

a) Define functions of R and n that calculate both of this quantities and check the identity

$$S_{n-1}(R) = \frac{dV_n(R)}{dR}$$

b) Using **ListLinePlot** and **Table**, plot the quantities $V_n(R)$ (in red color) and $S_n(R)$(in blue color) for $R = 1$ for the integer values of $n \in [1, 30]$ and fill the space in between those plots with yellow.

c) The Stirling's approximation to the Gamma function in this case is

$$\Gamma\left(\frac{n}{2}+1\right) \sim \sqrt{2\pi}e^{-n/2}\left(\frac{n}{2}\right)^{(n+1)/2}$$

Define a function $V_n^{(approx)}(R)$ that uses the Stirling's approximation and plot the ratio $\dfrac{V_n^{(approx)}(R)}{V_n(R)}$ for $n \in [20, 100]$ using **Plot**. Show that the limit of this ratio does not depend on the R value.

d) Use a Monte-Carlo method to evaluate the number π via the formula for the 6-dimensional ball. Call random numbers using **RandomReal** and 10^6 samples of 6-dimensional points. How accurate is this approximation?

Problem 3

Mandelbrot set
In this problem you will need to plot the Mandelbrot set again (like you did in MATLAB for homework 1).

a) Define a function **MandelbrotSlow** using **Module** and a **While** loop: the input parameters are x, y such that $z_0 = x+iy$ and the N - maximum number of iterations. The function should return the number of iterations for which z stays within a circle of radius r (use $r = 2$ as in homework1).

b) Before plotting anything with this function, you might notice it is very slow. Use **Compile** to define a new function **Mandelbrot** that is compiled (see Mathematica help on **Compile** for usage).

c) Function similar to **surf** in MATLAB is called **DensityPlot** in Mathematica. Use it to plot the function **Mandelbrot** on an interval $[-2, 0.6] \times [-1.3, 1.3]$ with $N = 100$. Make sure you use no mesh and an appropriate number of points for resolution. Use **GrayLevel** for **ColorFunction** option.

d) Now plot **Mandelbrot** on an interval $[-.3, 0.1] \times [.6, 1]$ with $N = 100$. Use **Hue** for **ColorFunction** option and modify it using the template (Hue[$a\#^b$]&) to find the parameters a, b that make this plot look nice to you (you can try $a = .6, b = .8$ for example).

e) Now plot **Mandelbrot** on an interval $[-0.0234, -0.0229] \times [0.99875, 0.99925]$ with $N = 100$. Observe the self-similarity property of this fractal structure.

f) Now plot **Mandelbrot** on an interval $[-0.75, -0.747] \times [0.06, 0.063]$ with $N = 1000$.This is a zoom near the boundary of the set (might take some time to compute).

g) Finally, create another function for a different map and submit the plot of it as well. Choose the color function and an appropriate axis limits

Problem 4

Solving equations
Solve the following equations in Mathematica

a) The quartic equation

$$z^4 + 3z^2 + 2z + 1 = 0$$

and output the solutions on the complex plane using **Graphics, Show** and **Point** along with the **ContourPlot** of the polynomial

b) Solve

$$\begin{aligned} x^3 + y^3 &= z \\ x + 2y &= 3z + 1 \\ xyz &\neq 0 \end{aligned}$$

where x, y, z are *modulo* 7

c) Find how many ways are there to pay \$3.54 postage with 10-, 23-, and 37-cent stamps with no change.

d) Linear algebra example. Solve the system of equations $Ax = b$ in Mathematica using multifrontal method and the Krylov subspace based method (see help of **LinearSolve** for details). Set $n = 50000$ and A to be a sparse matrix of size n with only values -2 on the main diagonal and 1 on the first super- and sub-diagonals (use the function **Band** from the lecture notes). Right hand side b should be a zero vector with one 1 at a random place. Which method is faster? What if you set the main diagonal values to -3?