Homework 2: Fun with Proofs

CSE 20 Introduction to Discrete Mathematics

Due 11am Monday July 14, 2014

The folowing exercises are taken from The Book of Proof

Exercises for Section 2.4 Biconditional Statements

Without changing their meanings, convert each of the following sentences into a sentence having the form *"P if and only if Q."*

2.4.1. For matrix A to be invertible, it is necessary and sufficient that $\det(A) \neq 0$

2.4.2. If a function has a constant derivative then it is linear, and conversely.

Exercises for Section 2.5 Truth Tables for Statements

A. Write a truth table for the logical statements.

2.5.3. $\sim(P \Rightarrow Q)$

2.5.7. $(P \wedge \sim P) \Rightarrow Q$

B. Write a truth table for the logical statements.

2.5.8. $P \vee (Q \wedge \sim R)$

Exercises for Section 2.6 Logical Equivalence

A. Use truth tables to show that the following statements are logically equivalent.

2.6.1. $P \wedge (Q \vee R) = (P \wedge Q) \vee (P \wedge R)$

2.6.3. $P \Rightarrow Q = (\sim P) \vee Q$

2.6.8. $\sim P \Leftrightarrow Q = (P \Rightarrow \sim Q) \wedge (\sim Q \Rightarrow P)$

B. Decide whether or not the following pairs of statements are logically equivalent.

2.6.14. $P \wedge (Q \vee \sim Q)$ and $(\sim P) \Rightarrow (Q \wedge \sim Q)$

Exercises for Section 2.7 Quantifiers

A. Write the following as an English sentence. Say whether it is true or false.

2.7.5. $\forall n \in \mathbb{N}, \exists X \in \mathscr{P}(\mathbb{N}), |X| < n$

Exercises for Section 2.9 Translating English to Symbolic Logic

A. Translate the following sentence into symbolic logic.

2.9.5 For every positive number ϵ, there is a positive number δ for which $|x - a| < \delta$ implies $|f(x) - f(a)| < \epsilon$

Exercises for Section 2.10 Negating Statements

2.10.1. Negate the following sentence.

- The number x is positive, but the number y is not positive.

Exercises for Chapter 4 Direct Proof

Use the method of direct proof to prove the following statements.

4.1. If x is an even integer, then x^2 is even.

4.2. If x is an odd integer, then x^3 is odd.

4.3. If a is an odd integer, then $a^2 + 3a + 5$ is odd.

4.4. Suppose $x, y \in \mathbb{Z}$. If x and y are odd, then the product xy is odd.

4.6. Suppose $a, b, c \in \mathbb{Z}$. If $a|b$ and $a|c$, then $a|(b+c)$.

4.8. Suppose a is an integer. If $5|2a$, then $5|a$

4.10. Suppose a and b are integers. If $a|b$, then $a|(3b^3 - b^2 + 5b)$.

4.12. If $x \in \mathbb{R}$ and $0 < x < 4$, then $\frac{4}{x(4-x)} \geq 1$.

4.13. Suppose $x, y \in \mathbb{R}$. If $x^2 + 5y = y^2 + 5x$, then $x = y$ or $x + y = 5$.

4.14. If $n \in \mathbb{Z}$, then $5n^2 + 3n + 7$ is odd. (Hint: Try cases.)

4.16. If two integers have the same parity, then their sum is even. (Hint: Try cases.)

4.18. Suppose x and y are positive real numbers. If $x < y$, then $x^2 < y^2$.

4.19. Suppose a, b and c are integers. If $a^2|b$ and $b^3|c$, then $a^6|c$.

4.20. If a is an integer and $a^2|a$, then $a \in \{-1, 0, 1\}$.

4.21. If p is prime and k is an integer for which $0 < k < p$, then p divides $\binom{p}{k} = \frac{p!}{k!(p-k)!}$.

4.23. If $n \in \mathbb{N}$, then $\binom{2n}{n} = \frac{(2n)!}{n!(2n-n)!}$ is even.

4.24. If $n \in \mathbb{N}$ and $n \geq 2$, then the numbers $n! + 2, n! + 3, n! + 4, n! + 5, \ldots, n! + n$ are all composite. (Thus for any $n \geq 2$ one can find n consecutive composite numbers. This means there are aribrarilty large "gaps" between prime numbers.)

4.26. Every odd integer is a difference of two squares. (Example $7 = 4^2 - 3^2$, $9 = 5^2 - 4^2$, etc.)

4.27. Suppose $a, b \in \mathbb{N}$. If $\gcd(a, b) > 1$, then $b|a$ or b is not prime.

4.28. If $a, b, c \in \mathbb{Z}$, then $c \cdot \gcd(a, b) \leq \gcd(ca, cb)$.

Exercises for Chapter 5 Contrapositive Proof

Prove the following statements using the method of contrapositive proof. *(In each case you should also think about how a direct proof would work. You will find in most cases that contrapositive is easier.)*

5.1. Suppose $n \in \mathbb{Z}$. If n^2 is even, then n is even.

5.2. Suppose $n \in \mathbb{Z}$. If n^2 is odd, then n is odd.

5.3. Suppose $a, b \in \mathbb{Z}$. If $a^2(b^2 - 2b)$ is odd, then a and b are odd.

5.4. Suppose $a, b, c, \in \mathbb{Z}$. If a does not divide bc, then a does not divide b.

5.5. Suppose $x \in \mathbb{R}$. If $x^2 + 5x < 0$, then $x < 0$.

5.6. Suppose $x \in \mathbb{R}$. If $x^3 - x > 0$, then $x > -1$.

5.9. Suppose $n \in \mathbb{Z}$. If $3 \nmid n^2$, then $3 \nmid n$.

5.10. Suppose $x, y, z \in \mathbb{Z}$ and $x \neq 0$. If $x \nmid yz$, then $x \nmid y$ and $x \nmid z$.

5.12. Suppose $a \in \mathbb{Z}$. If a^2 is not divisible by 4, then a is odd.

Prove the following statements using either direct or contrapositive proof. Sometimes one approach will be much easier than the other.

5.1.15. Suppose $x \in \mathbb{Z}$. If $x^3 - 1$ is even, then x is odd.

5.1.16. Suppose $x \in \mathbb{Z}$. If $x + y$ is even, then x and y have the same parity.

5.1.17. If n is odd , then $8|(n^2 - 1)$.

5.1.19. Suppose $a, b \in \mathbb{Z}$ and $n \in \mathbb{Z}$. If $a \equiv b(\mod n)$ and $a \equiv c(\mod n)$, then $c \equiv b(\mod n)$.

5.1.20. If $a \in \mathbb{Z}$ and $a \equiv 1(\mod 5)$, then $a^2 \equiv 1(\mod 5)$.

5.1.28. If $n \in \mathbb{Z}$, then $4 \nmid (n^2 - 3)$.