Homework 3: Groups and Proofs

CSE 20 Introduction to Discrete Mathematics

Due 2pm Tuesday (at the Loft) July 22, 2014 or in class on Monday 7/21

Groups

G.1. Give two reasons that the set of odd integers under addition is not a group.

G.2. Let $S = \mathbb{Z}$ be the set of all integers. Show that the binary operation of subtraction is not associative, therefore the set of integers under subtraction does *not* form a group.

G.3. Let $2\mathbb{Z}$ be the set of all even integers. Show that this forms a group under addition.

G.4. Let $S = \{1, 2, 3, 4, 5\}$. Is S a group under multiplication mod 6?

G.5. Show the set $S = \{1, 2, 3, 4, 5, 6\}$ is a group under multiplication mod 7.

G.6. Show the set $\mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$ is a group under addition mod 6.

G.7. For each $n > 1$, we define $U(n)$ to be the set of positive integers less than n that are relatively prime to n. For example $U(10) = \{1, 3, 7, 9\}$. Does $U(10)$ with multiplication mod 10 form a group?

G.8. Let $S = \{5, 15, 25, 35\}$. Is S a group under multiplication mod 40?

G.9. Show the subset $S = \{1, -1, i, -i\}$ of complex numbers $\mathbb{C}$ forms a group under complex multiplication $(a + bi)(c + di) = (ac - bd) + (ad + bc)i$.

G.10. Let $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. S forms a group under multiplication mod 11. Find the inverse of $8 \in S$. Find the inverse of $5 \in S$.

G.X1. (Extra Credit.) Let $S = \{1, 2, 3, \ldots, n - 1\}$. Prove that S a group under multiplication mod n if and only if n is prime.

G.X2. (Extra Credit.) Let $SL(2, F)$ denote the set of all 2×2 matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with determinant $ad - bc = 1$ mod 5 and entries $a, b, c, d \in F = \mathbb{Z}_5 = \{0, 1, 2, 3, 4\}$. Prove that $SL(2, F)$ forms a group under matrix multiplcation (with arithmatic performed mod 5). Find the inverse of the element $\begin{pmatrix} 3 & 4 \\ 4 & 4 \end{pmatrix}$. (Note you will need to find the inverse of a general element $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}_5)$ to show that $SL(2, \mathbb{Z}_5)$ is a group).

The folowing exercises are taken from The Book of Proof

Exercises for Chapter 6 Proof by Contradiction

A. Use the method of proof by contradiction to prove the following statements. (In each case, you should also think about how a direct or contrapositive proof would work. You will find in most cases that proof by contradiction is easier.

6.1. Suppose $n \in \mathbb{Z}$. If n is odd, then n^2 is odd.

6.2. Suppose $n \in \mathbb{Z}$. If n^2 is odd, then n is odd.

6.9. Suppose $a, b \in \mathbb{R}$. If a is rational and ab is irrational, then b is irrational.

6.10. There exist no integers a and b for which $21a + 30b = 1$.

6.15. If $b \in \mathbb{Z}$ and $b \nmid k$ for every $k \in \mathbb{N}$, then $b = 0$.

6.17. For every $n \in \mathbb{Z}, 4 \nmid (n^2 + 2)$.

The folowing exercises are taken from The Book of Proof

Exercises for Chapter 10 Mathematical Induction

Prove the following statements with either induction, strong induction or proof by smallest counterexample.

10.1. For every integer $n \in \mathbb{N}$, it follows that $1 + 2 + 3 + 4 + ... + n = \frac{n^2+n}{2}$.

10.3. For every integer $n \in \mathbb{N}$, it follows that $1^3 + 2^3 + 3^3 + 4^3 + ... + n^3 = \frac{n^2(n+1)^2}{4}$.

10.5. If $n \in \mathbb{N}$, then $2^1 + 2^2 + 2^3 + ... + 2^n = 2^{n+1} - 2$.

10.6. For every natural number n, it follows that $\sum_{i=1}^{n} (8i - 5) = 4n^2 - n$.

10.8. If $n \in \mathbb{N}$, then $\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + ... + \frac{n}{(n+1)!} = 1 - \frac{1}{(n+1)!}$.

10.12. For any integer $n \geq 0$, it follows that $9|(4^{3n} + 8)$.