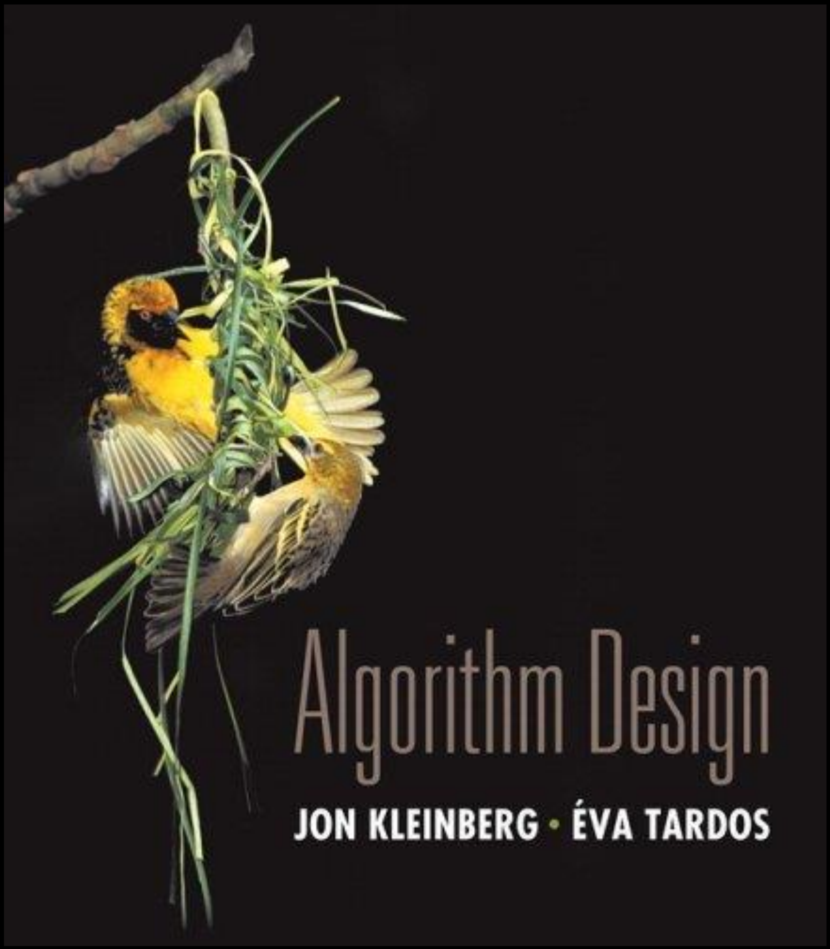

Chapter 1

Introduction: Some Representative Problems

(Continued)

Slides by Kevin Wayne.

Announcements

No class on Monday!

Please read Chapter 2 of Algorithm Design for Wednesday and Friday lectures!

Homework 1: Released sometime next week (hopefully) and due two weeks after release

Men's Preference Profile

	0th	1st	2nd	3rd	4th
Victor	Bertha	Amy	Diane	Erika	Clare
Wyatt	Diane	Bertha	Amy	Clare	Erika
Xavier	Bertha	Erika	Clare	Diane	Amy
Yancey	Amy	Diane	Clare	Bertha	Erika
Zeus	Bertha	Diane	Amy	Erika	Clare

Women's Preference Profile

	0th	1st	2nd	3rd	4th
Amy	Zeus	Victor	Wyatt	Yancey	Xavier
Bertha	Xavier	Wyatt	Yancey	Victor	Zeus
Clare	Wyatt	Xavier	Yancey	Zeus	Victor
Diane	Victor	Zeus	Yancey	Xavier	Wyatt
Erika	Yancey	Wyatt	Zeus	Xavier	Victor

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Clare	Wyatt	Xavier	Yancey	Zeus	Victor
Diane	Victor	Zeus	Yancey	Xavier	Wyatt
Erika	Yancey	Wyatt	Zeus	Xavier	Victor

What if Women Propose?

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Diane	Victor	Zeus	Yancey	Xavier	Wyatt
Erika	Yancey	Wyatt	Zeus	Xavier	Victor

Comparison of Men and Women Proposing

Men propose

Men's Preference Profile

	0^{th}	1^{st}	2^{nd}	3^{rd}	4^{th}
Victor	Bertha	Amy	Diane	Erika	Clare
Wyatt	Diane	Bertha	Amy	Clare	Erika
Xavier	Bertha	Erika	Clare	Diane	Amy
Yancey	Amy	Diane	Clare	Bertha	Erika
Zeus	Bertha	Diane	Amy	Erika	Clare

Women's Preference Profile

	0^{th}	1^{st}	2^{nd}	3^{rd}	4^{th}
Amy	Zeus	Victor	Wyatt	Yancey	Xavier
Bertha	Xavier	Wyatt	Yancey	Victor	Zeus
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Diane	Victor	Zeus	Yancey	Xavier	Wyatt
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Women propose

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Erika	Yancey	Wyatt	Zeus	Xavier	Victor

Understanding the Solution

Q. For a given problem instance, there may be several stable matchings. Do all executions of Gale-Shapley yield the same stable matching? If so, which one?

An instance with two stable matchings.

- A-X, B-Y, C-Z.
- A-Y, B-X, C-Z.

	1st	2nd	3rd
Xavier	A	B	C
Yancey	B	A	C
Zeus	A	B	C

	1st	2nd	3rd
Amy	Y	X	Z
Bertha	X	Y	Z
Clare	X	Y	Z

Understanding the Solution

Q. For a given problem instance, there may be several stable matchings. Do all executions of Gale-Shapley yield the same stable matching? If so, which one?

Def. Man m is a valid partner of woman w if there exists some stable matching in which they are matched.

Man-optimal assignment. Each man receives best valid partner.

Claim. All executions of GS yield man-optimal assignment, which is a stable matching!

- No reason a priori to believe that man-optimal assignment is perfect, let alone stable.
- Simultaneously best for each and every man.

Man Optimality

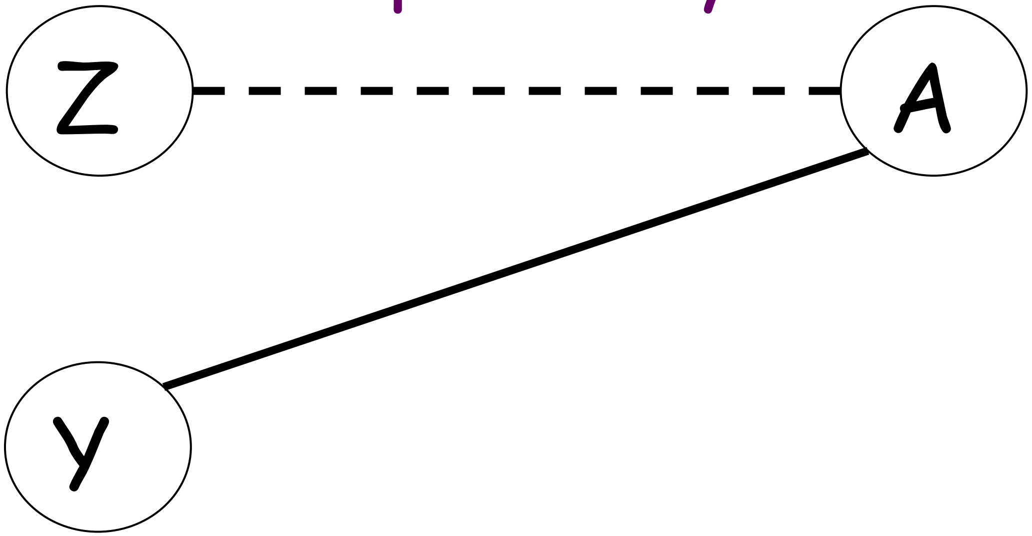
Claim. GS matching S* is man-optimal.

Pf. (by contradiction)

- Suppose some man is paired with someone other than best partner. Men propose in decreasing order of preference ⇒ some man is rejected by valid partner.
- Let Y be first such man, and let A be first valid woman that rejects him.
- Let S be a stable matching where A and Y are matched.
- When Y is rejected, A forms (or reaffirms) engagement with a man, say Z, whom she prefers to Y.
- Let B be Z's partner in S.
- Z not rejected by any valid partner at the point when Y is rejected by A. Thus, Z prefers A to B.
- But A prefers Z to Y.
- Thus A-Z is unstable in S. ▪

since this is first rejection by a valid partner

Man Optimality Proof



- Assume A "dumping" (or ignoring) Z in favor of Y is the 1st time a man is dumped/ignored by a valid match
  - Z-A must be a stable match to be valid
- A must be Y's most preferred possibly valid match (or this would not be the first "dumping" by a valid match!)
- A must prefer Y to Z or would not have "dumped" Z
- Thus, Z-A is not a stable match (therefore, Z-A not a valid match → contradiction of the assumption!)

Stable Matching Summary

Stable matching problem. Given preference profiles of n men and n women, find a stable matching.

no man and woman prefer to be with each other than assigned partner

Gale-Shapley algorithm. Finds a stable matching in $O(n^2)$ time.

Man-optimality. In version of GS where men propose, each man receives best valid partner.

w is a valid partner of m if there exist some stable matching where m and w are paired

Q. Does man-optimality come at the expense of the women?

Woman Pessimality

Woman-pessimal assignment. Each woman receives worst valid partner.

Claim. GS finds woman-pessimal stable matching S*.

Pf.

- Suppose A-Z matched in S*, but Z is not worst valid partner for A.
- There exists stable matching S in which A is paired with a man, say Y, whom she likes less than Z.
- Let B be Z's partner in S.
- Z prefers A to B.
- Thus, A-Z is an unstable in S. ▪

Extensions: Matching Residents to Hospitals

Ex: Men ≈ hospitals, Women ≈ med school residents.

Variant 1. Some participants declare others as unacceptable.

resident A unwilling to work in Cleveland

Variant 2. Unequal number of men and women.

Variant 3. Limited polygamy.

hospital X wants to hire 3 residents

Def. Matching S unstable if there is a hospital h and resident r such that:

- h and r are acceptable to each other; and
- either r is unmatched, or r prefers h to her assigned hospital; and
- either h does not have all its places filled, or h prefers r to at least one of its assigned residents.

Application: Matching Residents to Hospitals

NRMP. (National Resident Matching Program)

- Original use just after WWII. ← predates computer usage
- Ides of March, 23,000+ residents.

Rural hospital dilemma.

- Certain hospitals (mainly in rural areas) were unpopular and declared unacceptable by many residents.
- Rural hospitals were under-subscribed in NRMP matching.
- How can we find stable matching that benefits "rural hospitals"?

Rural Hospital Theorem. Rural hospitals get exactly same residents in every stable matching!

Lessons Learned

Powerful ideas learned in course.

- Isolate underlying structure of problem.
- Create useful and efficient algorithms.

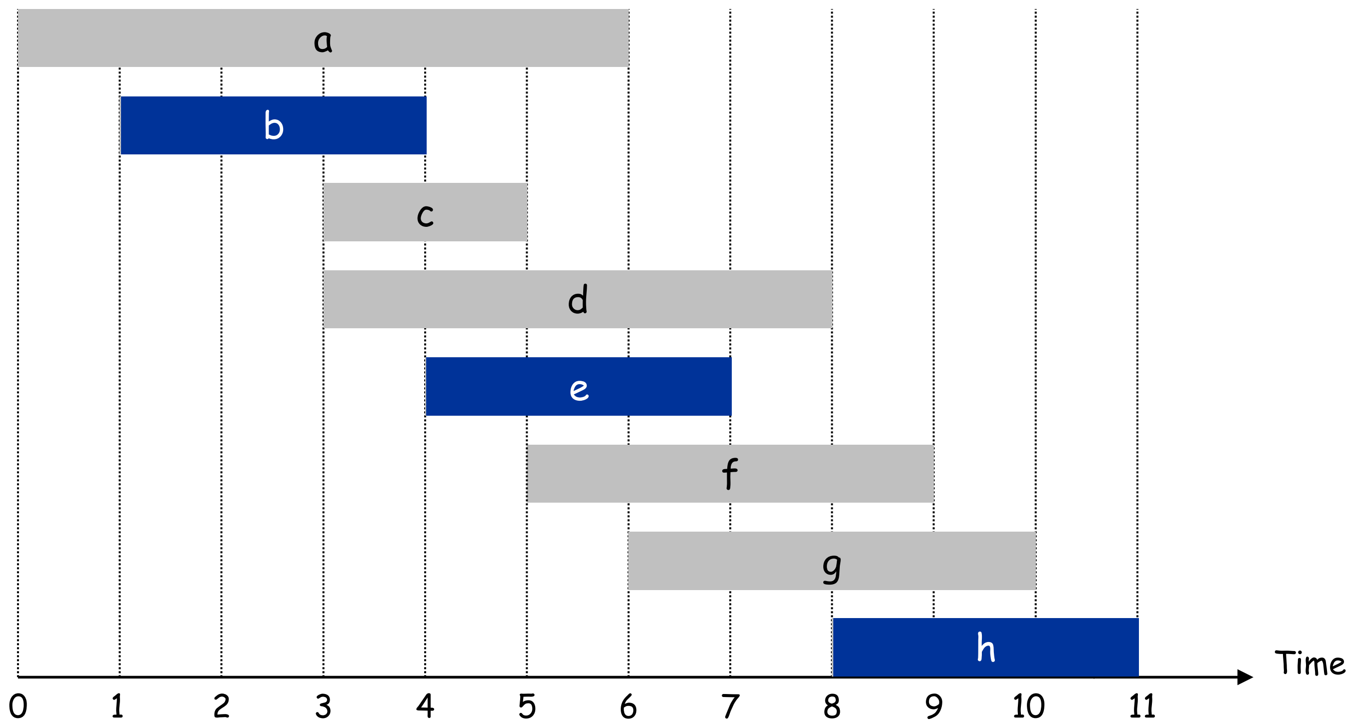
Potentially deep social ramifications. [legal disclaimer]

1.2 Five Representative Problems

Interval Scheduling

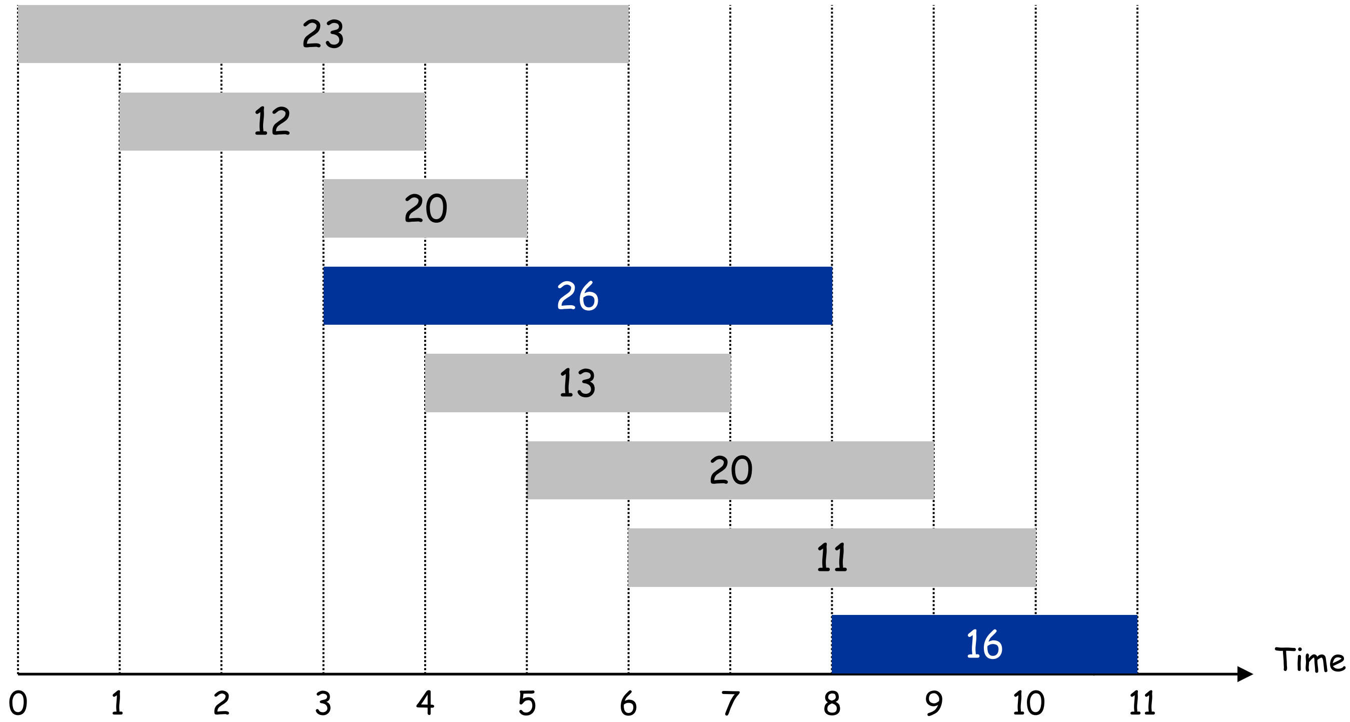
Input. Set of jobs with start times and finish times.
Goal. Find maximum cardinality subset of mutually compatible jobs.

jobs don't overlap



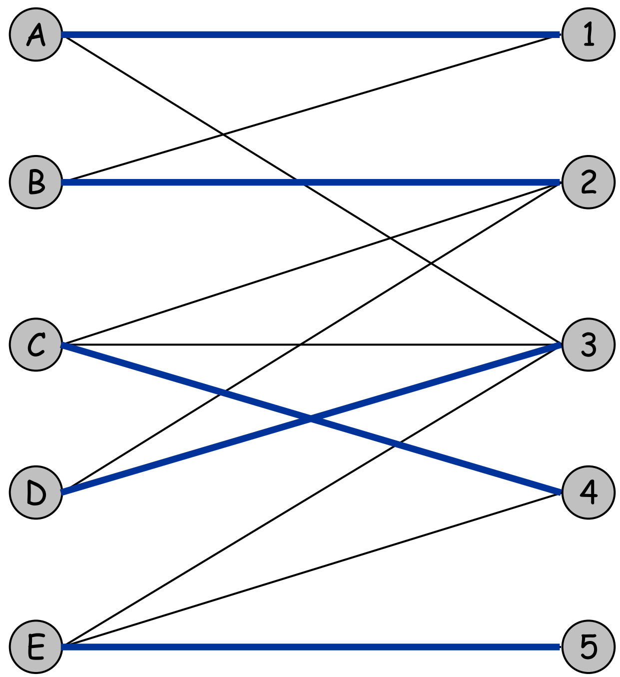
Weighted Interval Scheduling

Input. Set of jobs with start times, finish times, and weights.
Goal. Find maximum weight subset of mutually compatible jobs.



Bipartite Matching

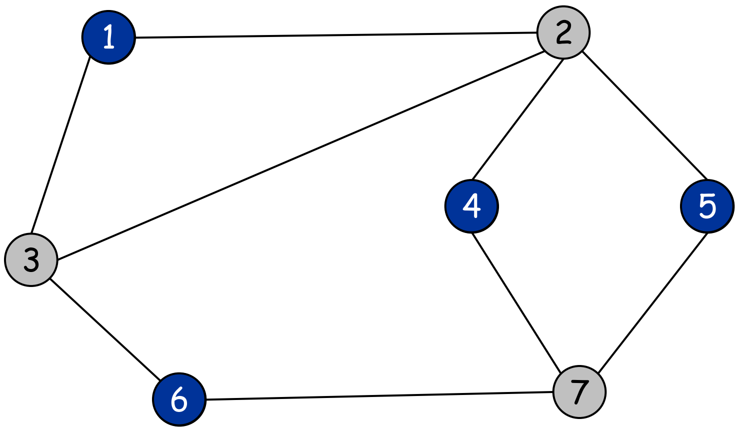
Input. Bipartite graph.
Goal. Find maximum cardinality matching.



Independent Set

Input. Graph.
Goal. Find maximum cardinality independent set.

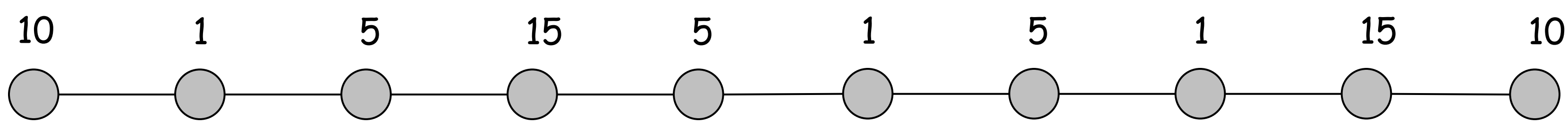
subset of nodes such that no two joined by an edge



Competitive Facility Location

Input. Graph with weight on each node.
Game. Two competing players alternate in selecting nodes. Not allowed to select a node if any of its neighbors have been selected.

Goal. Select a maximum weight subset of nodes.



Second player can guarantee 20, but not 25.

Which problems are harder than others?

Representative problems

- Interval scheduling
- Weighted interval scheduling
- Bipartite matching
- Independent set
- Competitive facility location

How can we <u>prove</u> it?

<u>**Reductions**</u>!
If one problem can be expressed as another problem*

* "Re-expressing" the problem must not require more time complexity than an algorithm for solving the original problem

Five Representative Problems

Variations on a theme: independent set.

Interval scheduling: n log n greedy algorithm.

Weighted interval scheduling: n log n dynamic programming algorithm.

Bipartite matching: n^k max-flow based algorithm.

Independent set: NP-complete.

Competitive facility location: PSPACE-complete.