Virtual Work Method (Unit Load Method – Leet Ch. 10)

The most general method for finding displacements and rotations. Can be used to find deflection and/or rotation in any type of structure due to any cause.

P-System (real): Actual loads, temperature changes, fabrication errors, or support movements cause internal forces (moments) and deflections.
Q-System (imaginary): A unit load (moment) is applied on the pristine bare structure and generates internal forces (moments) and deflections.

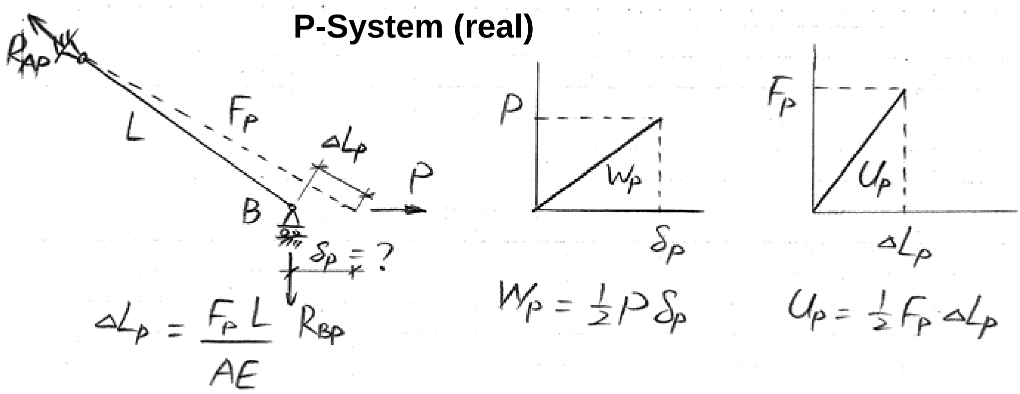
The Q-system is created as follows:

a. A unit load is applied at the point on the pristine bare structure where deflection is to be calculated, or a unit moment is applied where rotation is to be calculated.
b. Support reactions are generated due to the unit load or moment.
c. Internal member forces (moments) occur due to the unit load or moment.

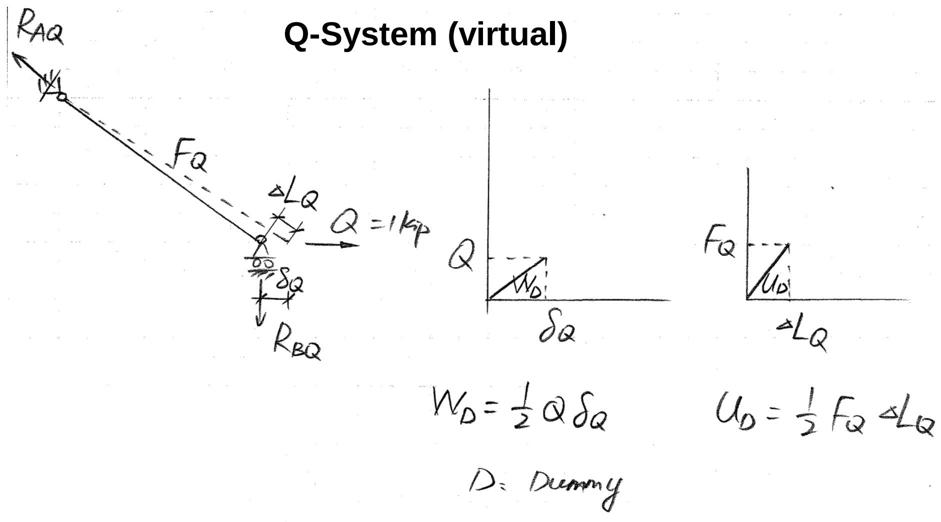
The virtual work method is produced by an imaginary two-step process:

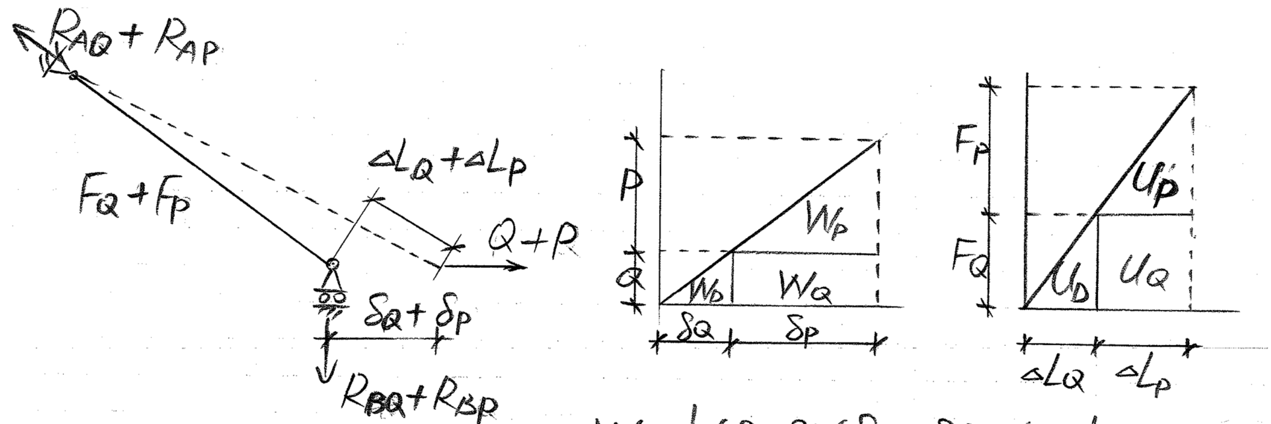
1. Image the unit load(s) is applied to the pristine bare structure first, i.e. Q-system is generated first.
2. On top of the unit load(s), the real loads and other conditions are added to the structure so that the structure deforms furthermore. When this further deformation occurs, components of the initially applied Q-system do work.

P-System (real)



Q-System (virtual)





$$W=\frac{1}{2}(P+Q)(\delta_Q+\delta_P) \quad U=\frac{1}{2}(F_P+F_Q)\cdot(\Delta L_P+\Delta L_Q)$$

$$W=U \Rightarrow W_Q=U_Q \Rightarrow Q\,\delta_P=F_Q\cdot\Delta L_P$$

Again, the virtual work method is produced by an imaginary two-step process:

1. Image the unit load is applied to the pristine bare structure first
2. On top of the unit load, the real loads are added onto the structure …

<u>DURING STEP-2</u>:

W_Q: External virtual work done by the unit load (and its reaction) as it moves through the <u>real</u> displacement (and support settlement).

U_Q: Virtual strain energy, i.e. work done by virtual internal forces (caused by unit load) as they move through <u>real</u> deformations (caused by actual loads).

In general, $W_Q = \sum Q\delta_P$ includes work done by the unit load, and work done by reactions in Q-System acting through support settlements in P-System.

- $+\delta_P$ is in the same direction as the dummy unit load (moment).
- A pair of unit loads can be used to find relative displacement (Leet, Exmp. 10.7).

Causes of displacement in P-System:

- Applied loads
- Temperature
- Fabrication error
- Support movements (settlements)

Effects contributing to virtual strain energy U_Q: axial, bending, torsion, and shear deformations in the P-System:

$$W_Q = U_Q = \sum\left(\int F_Q\, d\Delta L_P + \int M_Q\, d\theta_P + \int T_Q\, d\phi_P + \int V_Q\, d\delta_P\right)$$
$$= \sum\left(F_Q \Delta L_P + \int \frac{M_Q M_P}{EI} dx + \int \frac{T_Q T_P}{GJ} dx + \int \frac{f V_Q V_P}{GA} dx\right)$$

Integral $\int_{x=0}^{L} \ldots dx$ is along the member length.

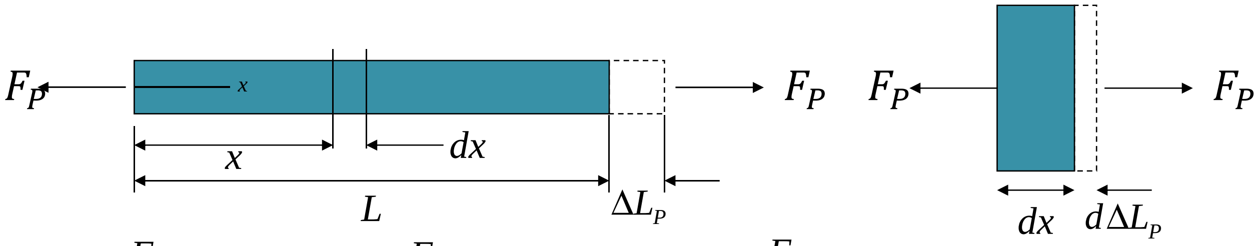
Axial Effects

$$W_Q = U_Q = \sum \left(\int F_Q \boxed{d\Delta L_P} + \int M_Q d\theta_P + \int T_Q d\phi_P + \int V_Q d\delta_P \right)$$

F_Q: axial force in a member due to the unit load in Q-System (const. along member)
Oftentimes, axial deformation $d\Delta L_P$ is uniform along the entire member length:

$$\int F_Q d\Delta L_P = F_Q \boxed{\Delta L_P}$$

- $+F_Q$ if tension; $+\Delta L_P$ if elongation
- P-System with temperature change: $\Delta L_P = \Delta t \cdot \alpha \cdot L$
- P-System with fabrication error: ΔL_P = fabr. error in length (+ if too long)
- P-System with loads causing axial force F_P :



$$\sigma_x = \frac{F_P}{A} \qquad \varepsilon_x = \frac{\sigma_x}{E} = \frac{F_P}{EA} \qquad d\Delta L_P = \varepsilon_x dx = \frac{F_P}{EA} dx$$

If const. F_P along the member: $\Delta L_P = \int_{x=0}^{L} d\Delta L_P = \frac{F_P L}{EA}$

Flexural Effects

$$W_Q = U_Q = \sum \left(\int F_Q d\Delta L_P + \int M_Q \boxed{d\theta_P} + \int T_Q d\phi_P + \int V_Q d\delta_P \right)$$

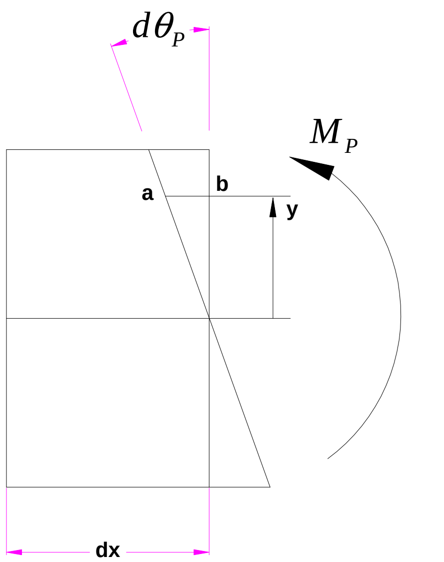
- For the axial fiber with height *y* from neutral axis:

$$\sigma_x = -\frac{M_P y}{I_z} \qquad \varepsilon_x = \frac{\sigma_x}{E} = \frac{M_P y}{EI_z}$$

- Deformation of the fiber:

$$\overline{ab} = -\varepsilon_x dx$$

$$d\theta_P = \frac{-\varepsilon_x dx}{y} = \boxed{\frac{M_P}{EI_z} dx}$$



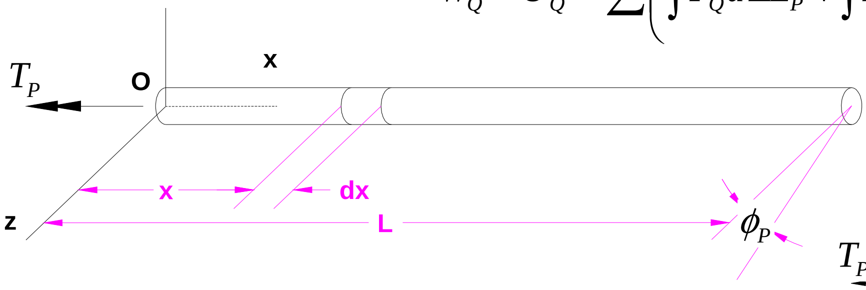
- a.k.a, moment-curvature formula:

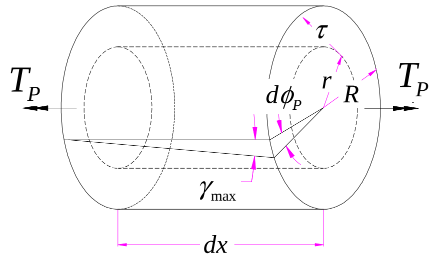
$$\psi = \frac{d\theta_P}{dx} = \frac{M_P}{EI_z}$$

- M_Q and M_P are usually not const. along the member (unlike axial forces F_Q and F_P)
- Use same sign convention for M_Q and M_P

Torsional Effects

$$W_Q = U_Q = \sum\left(\int F_Q d\Delta L_P + \int M_Q d\theta_P + \int T_Q d\phi_P + \int V_Q d\delta_P\right)$$





Shear stress at distance *r* from center:

$$\tau = \frac{T_P r}{J} \qquad T_P = \text{Torque}; \quad r = \text{Radius};$$

J = Torsional constant. If section is solid circular, J = polar moment of inertia
For other section shapes, see handout Mechanics of Materials, Ch3.10 and 3.11

Shear strain: $\gamma = \frac{\tau}{G} = \frac{T_P r}{GJ} \qquad \gamma_{max} = \frac{T_P R}{GJ} \qquad d\phi_P = \frac{\gamma_{max}}{R} dx = \frac{T_P}{GJ} dx$

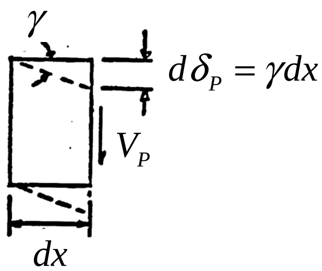
Shear modulus: $G = \frac{E}{2(1+\nu)}$, ν = Poisson's ratio

If T_P is const. along the member: $\phi_P = \int_0^L d\phi_P = \int_0^L \frac{T_P}{GJ} dx = \frac{T_P L}{GJ}$

Courtesy of Prof. David W. Scott, CEE, Gatech

Shear Effects

$$W_Q = U_Q = \sum\left(\int F_Q d\Delta L_P + \int M_Q d\theta_P + \int T_Q d\phi_P + \int V_Q d\delta_P\right)$$



In P-System: $d\delta_P = \gamma dx = \frac{\tau}{G} dx = \frac{f V_P}{GA} dx$

Form factor:

f = { 1.2 for rectangular section; 10/9 for circular section; 1.12 for wide flange

Virtual work – combined effects

$$\sum Q\delta_P = \sum\left(F_Q \Delta L_P + \int \frac{M_Q M_P}{EI} dx + \int \frac{T_Q T_P}{GJ} dx + \int \frac{f V_Q V_P}{GA} dx\right)$$

Including support movements

Length change induced by temperature and fabrication errors