4.5. Member Stiffness Relations in Global Coordinates

(Kassimali §3.6)

- Local stiffness relations

$$\mathbf{Q} = \mathbf{ku}$$

- Local-global force transformation

$$\mathbf{F} = \mathbf{T}^T\mathbf{Q} = \mathbf{T}^T\mathbf{ku}$$

- Substitute displacement transformation $\mathbf{u} = \mathbf{Tv}$ into the above:

$$\mathbf{F} = \mathbf{T}^T\mathbf{kTv}$$

- Defining *member stiffness matrix in the global coordinate system*:

$$\mathbf{K} = \mathbf{T}^T\mathbf{kT}$$

- We have

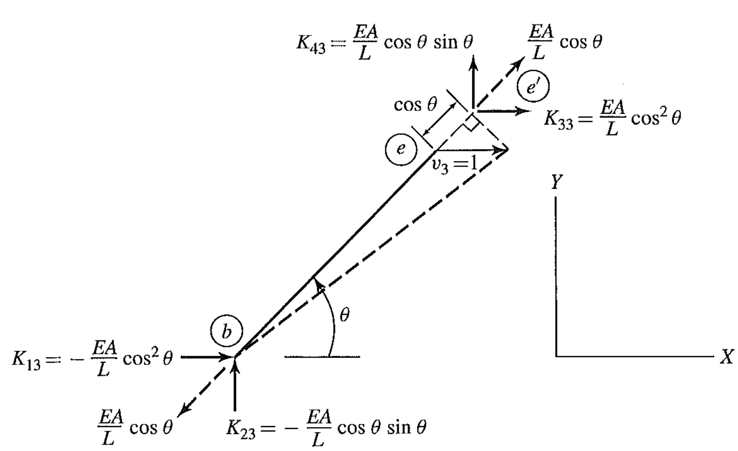
$$\mathbf{F} = \mathbf{Kv}$$

- Matrix $\mathbf{K}$ is symmetric (like $\mathbf{k}$):

$$\mathbf{K} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\theta & -\sin\theta \\ 0 & 0 & \sin\theta & \cos\theta \end{bmatrix} \frac{EA}{L} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \\ 0 & 0 & -\sin\theta & \cos\theta \end{bmatrix}$$

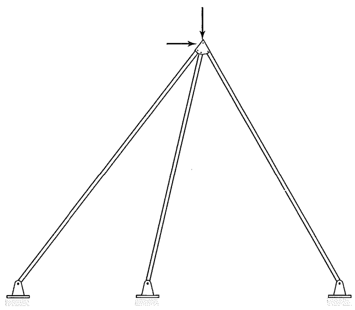
$$\mathbf{K} = \frac{EA}{L} \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta & -\cos^2\theta & -\cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta & -\cos\theta\sin\theta & -\sin^2\theta \\ -\cos^2\theta & -\cos\theta\sin\theta & \cos^2\theta & \cos\theta\sin\theta \\ -\cos\theta\sin\theta & -\sin^2\theta & \cos\theta\sin\theta & \sin^2\theta \end{bmatrix}$$

a stiffness coefficient K_{ij} represents the force at the location and in the direction of F_i required, along with other end forces, to cause a unit value of displacement v_j, while all other end displacements are zero. Thus, the jth column of matrix $\mathbf{K}$ consists of the end forces in the global coordinate system required to cause a unit value of the end displacement v_j, while all other end displacements are zero.

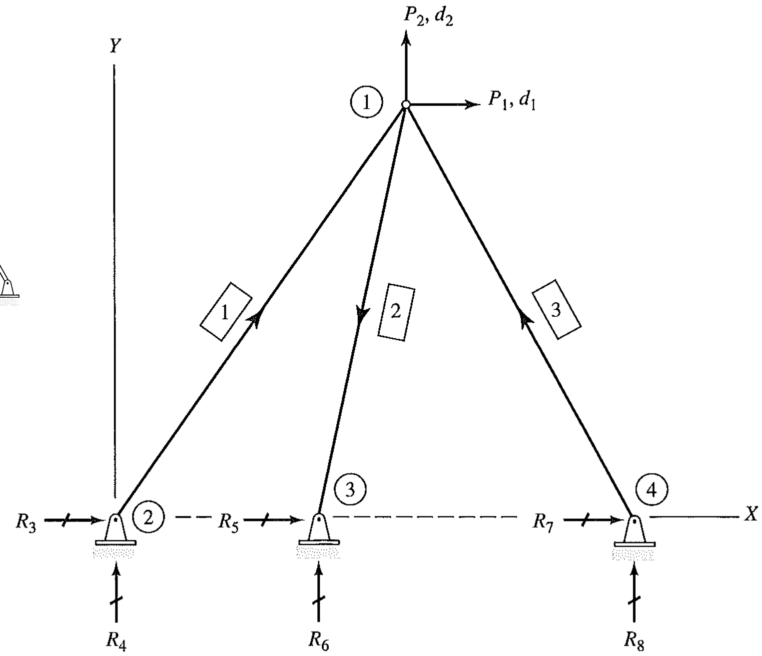


(c) Third Column of $\mathbf{K}$ ($v_3 = 1$, $v_1 = v_2 = v_4 = 0$)

4.6. Structure Stiffness Relations (Kassimali §3.6)



(a) Truss



(b) Analytical Model

Equilibrium Equations

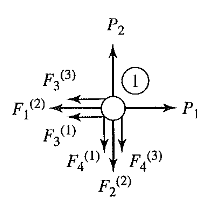
$$P_1 = F_3^{(1)} + F_1^{(2)} + F_3^{(3)}$$
$$P_2 = F_4^{(1)} + F_2^{(2)} + F_4^{(3)}$$

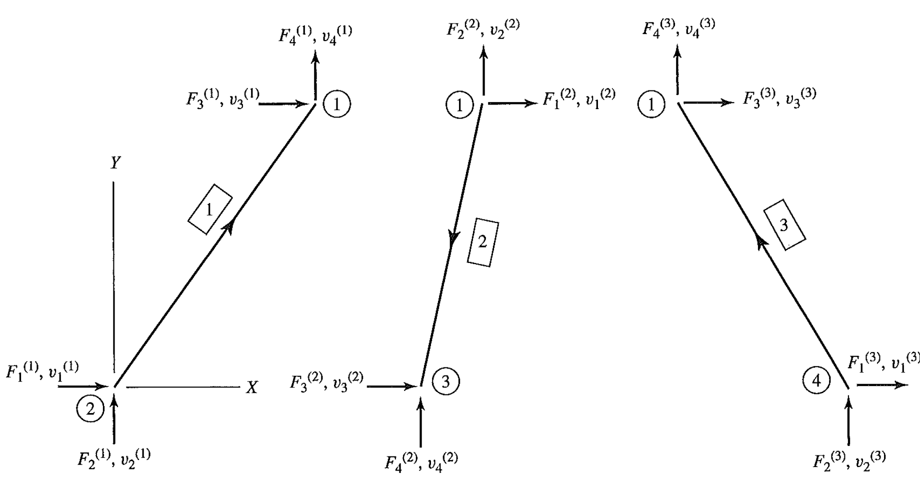
Compatibility Equations

$$v_1^{(1)} = v_2^{(1)} = 0 \qquad v_3^{(1)} = d_1 \qquad v_4^{(1)} = d_2$$

$$v_1^{(2)} = d_1 \qquad v_2^{(2)} = d_2 \qquad v_3^{(2)} = v_4^{(2)} = 0$$

$$v_1^{(3)} = v_2^{(3)} = 0 \qquad v_3^{(3)} = d_1 \qquad v_4^{(3)} = d_2$$





Equilibrium Equations

$$P_1 = F_3^{(1)} + F_1^{(2)} + F_3^{(3)}$$
$$P_2 = F_4^{(1)} + F_2^{(2)} + F_4^{(3)}$$

Compatibility Equations

$$v_1^{(1)} = v_2^{(1)} = 0 \qquad v_3^{(1)} = d_1 \qquad v_4^{(1)} = d_2$$
$$v_1^{(2)} = d_1 \qquad v_2^{(2)} = d_2 \qquad v_3^{(2)} = v_4^{(2)} = 0$$
$$v_1^{(3)} = v_2^{(3)} = 0 \qquad v_3^{(3)} = d_1 \qquad v_4^{(3)} = d_2$$

- Member 1:

$$\begin{bmatrix} F_1^{(1)} \\ F_2^{(1)} \\ F_3^{(1)} \\ F_4^{(1)} \end{bmatrix} = \begin{bmatrix} K_{11}^{(1)} & K_{12}^{(1)} & K_{13}^{(1)} & K_{14}^{(1)} \\ K_{21}^{(1)} & K_{22}^{(1)} & K_{23}^{(1)} & K_{24}^{(1)} \\ K_{31}^{(1)} & K_{32}^{(1)} & K_{33}^{(1)} & K_{34}^{(1)} \\ K_{41}^{(1)} & K_{42}^{(1)} & K_{43}^{(1)} & K_{44}^{(1)} \end{bmatrix} \begin{bmatrix} v_1^{(1)} \\ v_2^{(1)} \\ v_3^{(1)} \\ v_4^{(1)} \end{bmatrix}$$

$$F_3^{(1)} = K_{31}^{(1)} v_1^{(1)} + K_{32}^{(1)} v_2^{(1)} + K_{33}^{(1)} v_3^{(1)} + K_{34}^{(1)} v_4^{(1)} = K_{33}^{(1)} d_1 + K_{34}^{(1)} d_2$$
$$F_4^{(1)} = K_{41}^{(1)} v_1^{(1)} + K_{42}^{(1)} v_2^{(1)} + K_{43}^{(1)} v_3^{(1)} + K_{44}^{(1)} v_4^{(1)} = K_{43}^{(1)} d_1 + K_{44}^{(1)} d_2$$

- Member 2:

$$F_1^{(2)} = K_{11}^{(2)} d_1 + K_{12}^{(2)} d_2$$
$$F_2^{(2)} = K_{21}^{(2)} d_1 + K_{22}^{(2)} d_2$$

- Member 3:

$$F_3^{(3)} = K_{33}^{(3)} d_1 + K_{34}^{(3)} d_2$$
$$F_4^{(3)} = K_{43}^{(3)} d_1 + K_{44}^{(3)} d_2$$

- Structure stiffness relations between joint loads **P** and jot displacements **d**:

$$P_1 = F_3^{(1)} + F_1^{(2)} + F_3^{(3)} = \left(K_{33}^{(1)} + K_{11}^{(2)} + K_{33}^{(3)}\right) d_1 + \left(K_{34}^{(1)} + K_{12}^{(2)} + K_{34}^{(3)}\right) d_2$$
$$P_2 = F_4^{(1)} + F_2^{(2)} + F_4^{(3)} = \left(K_{43}^{(1)} + K_{21}^{(2)} + K_{43}^{(3)}\right) d_1 + \left(K_{44}^{(1)} + K_{22}^{(2)} + K_{44}^{(3)}\right) d_2$$

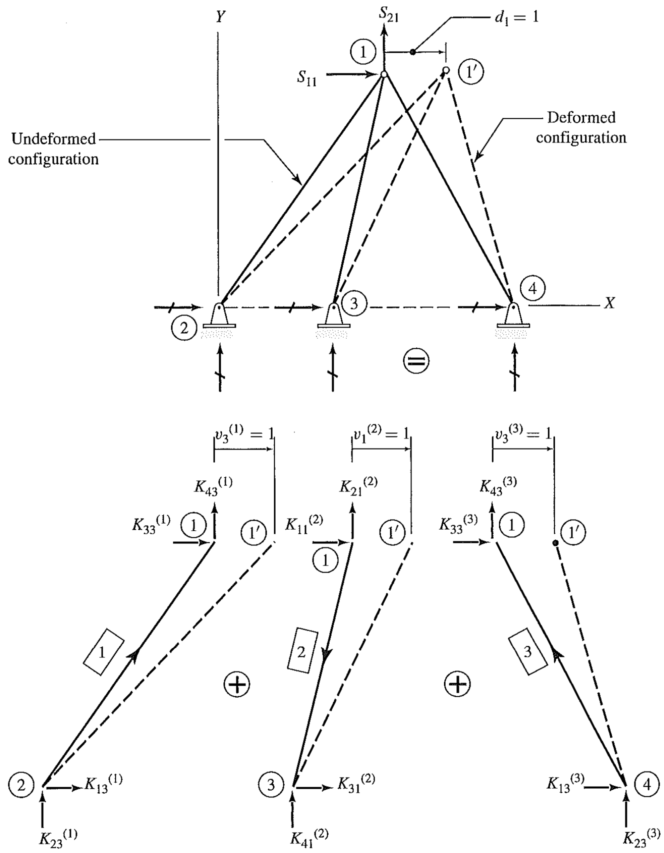
$$\boxed{\mathbf{P} = \mathbf{S}\mathbf{d}}$$

- Stiffness matrix **S**:

$$\mathbf{S} = \begin{bmatrix} K_{33}^{(1)} + K_{11}^{(2)} + K_{33}^{(3)} & K_{34}^{(1)} + K_{12}^{(2)} + K_{34}^{(3)} \\ K_{43}^{(1)} + K_{21}^{(2)} + K_{43}^{(3)} & K_{44}^{(1)} + K_{22}^{(2)} + K_{44}^{(3)} \end{bmatrix}$$

- Stiffness matrix of a linear elastic structure is always symmetric

The structure stiffness matrix **S** can be interpreted in a manner analogous to the member stiffness matrix. *A structure stiffness coefficient S_{ij} represents the force at the location and in the direction of P_i required, along with other joint forces, to cause a unit value of the displacement d_j, while all other joint displacements are zero.* Thus, the jth column of the structure stiffness matrix **S** consists of the joint loads required, at the locations and in the directions of all the degrees of freedom of the structure, to cause a unit value of the displacement d_j while all other displacements are zero.

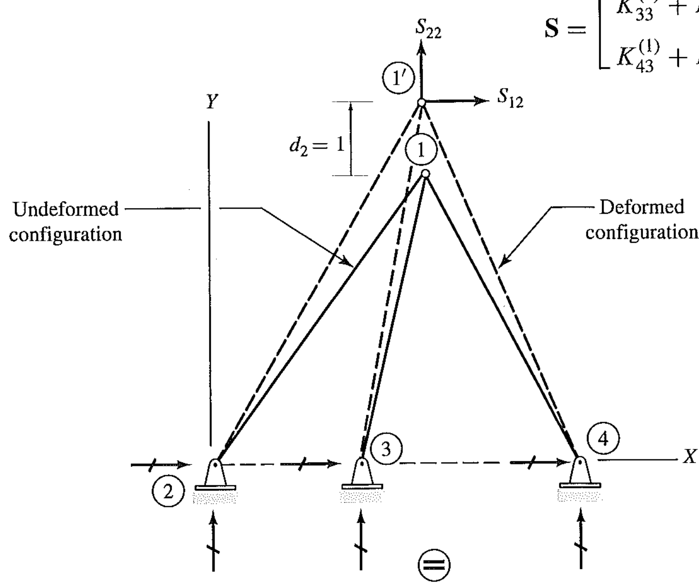


(a) First Column of **S** ($d_1 = 1, d_2 = 0$)

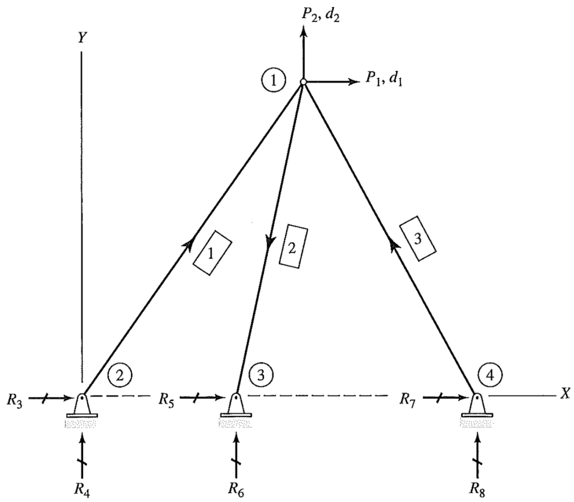
$$\mathbf{S} = \begin{bmatrix} K_{33}^{(1)} + K_{11}^{(2)} + K_{33}^{(3)} & \cdots \\ K_{43}^{(1)} + K_{21}^{(2)} + K_{43}^{(3)} & \end{bmatrix}$$

A structure stiffness coefficient S_{ij} represents the force at the location and in the direction of P_i required, along with other joint forces, to cause a unit value of the displacement d_j, while all other joint displacements are zero.

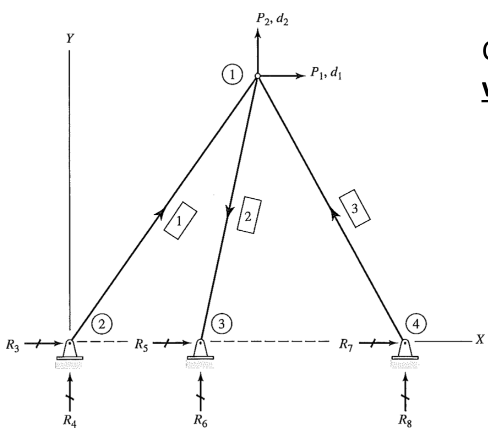
$$\mathbf{S} = \begin{bmatrix} K_{33}^{(1)} + K_{11}^{(2)} + K_{33}^{(3)} & K_{34}^{(1)} + K_{12}^{(2)} + K_{34}^{(3)} \\ K_{43}^{(1)} + K_{21}^{(2)} + K_{43}^{(3)} & K_{44}^{(1)} + K_{22}^{(2)} + K_{44}^{(3)} \end{bmatrix}$$



Assembly Using Member Code Numbers



- Code numbers for every member are the numbers of the structure DOFs or restrained coordinates:
  - v_1 – X direction of beginning joint
  - v_2 – Y direction of beginning joint
  - v_3 – X direction of end joint
  - v_4 – Y direction of end joint
- Member 1: 3, 4, 1, 2
- Member 2: 1, 2, 5, 6
- Member 3: 7, 8, 1, 2



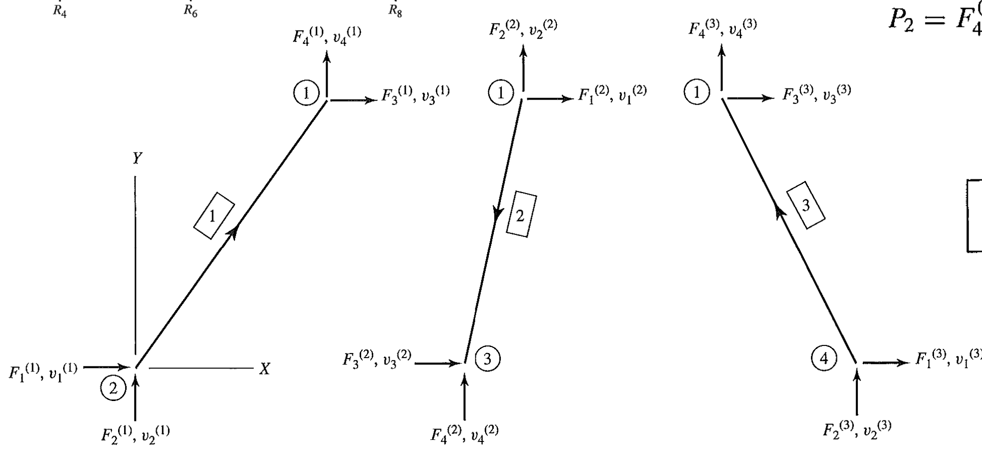
Code numbers of concern during assembly of **<u>joint load vector</u>**: 1 ~ *NDOF*

$$\mathbf{F}_1 = \begin{array}{cc} \begin{bmatrix} F_1^{(1)} \\ F_2^{(1)} \\ F_3^{(1)} \\ F_4^{(1)} \end{bmatrix} & \begin{array}{c} 3 \\ 4 \\ 1 \\ 2 \end{array} \end{array} \quad \mathbf{F}_2 = \begin{array}{cc} \begin{bmatrix} F_1^{(2)} \\ F_2^{(2)} \\ F_3^{(2)} \\ F_4^{(2)} \end{bmatrix} & \begin{array}{c} 1 \\ 2 \\ 5 \\ 6 \end{array} \end{array} \quad \mathbf{F}_3 = \begin{array}{cc} \begin{bmatrix} F_1^{(3)} \\ F_2^{(3)} \\ F_3^{(3)} \\ F_4^{(3)} \end{bmatrix} & \begin{array}{c} 7 \\ 8 \\ 1 \\ 2 \end{array} \end{array}$$

$$P_1 = F_3^{(1)} + F_1^{(2)} + F_3^{(3)}$$
$$P_2 = F_4^{(1)} + F_2^{(2)} + F_4^{(3)}$$

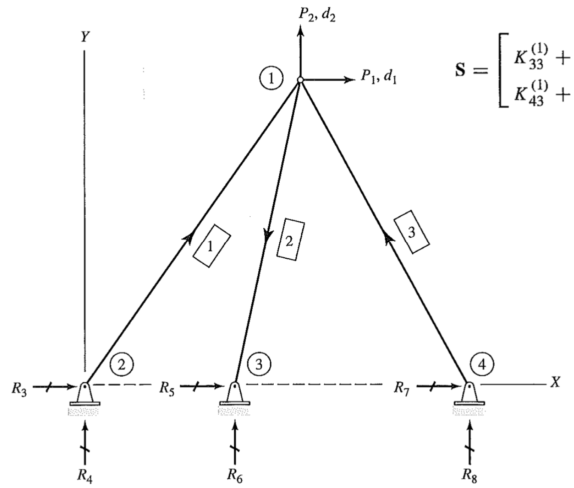
$$\mathbf{P} = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}$$

$$\boxed{\mathbf{P} = \mathbf{S}\mathbf{d}}$$



(b) Member End Forces and End Displacements in the Global Coordinate System

$$\mathbf{K}_1 = \begin{array}{c} \begin{array}{cccc} 3 & 4 & 1 & 2 \end{array} \\ \left[\begin{array}{cccc} K_{11}^{(1)} & K_{12}^{(1)} & K_{13}^{(1)} & K_{14}^{(1)} \\ K_{21}^{(1)} & K_{22}^{(1)} & K_{23}^{(1)} & K_{24}^{(1)} \\ K_{31}^{(1)} & K_{32}^{(1)} & K_{33}^{(1)} & K_{34}^{(1)} \\ K_{41}^{(1)} & K_{42}^{(1)} & K_{43}^{(1)} & K_{44}^{(1)} \end{array}\right] \end{array} \begin{array}{c} \\ 3 \\ 4 \\ 1 \\ 2 \end{array} \quad \mathbf{K}_2 = \begin{array}{c} \begin{array}{cccc} 1 & 2 & 5 & 6 \end{array} \\ \left[\begin{array}{cccc} K_{11}^{(2)} & K_{12}^{(2)} & K_{13}^{(2)} & K_{14}^{(2)} \\ K_{21}^{(2)} & K_{22}^{(2)} & K_{23}^{(2)} & K_{24}^{(2)} \\ K_{31}^{(2)} & K_{32}^{(2)} & K_{33}^{(2)} & K_{34}^{(2)} \\ K_{41}^{(2)} & K_{42}^{(2)} & K_{43}^{(2)} & K_{44}^{(2)} \end{array}\right] \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 5 \\ 6 \end{array} \quad \mathbf{K}_3 = \begin{array}{c} \begin{array}{cccc} 7 & 8 & 1 & 2 \end{array} \\ \left[\begin{array}{cccc} K_{11}^{(3)} & K_{12}^{(3)} & K_{13}^{(3)} & K_{14}^{(3)} \\ K_{21}^{(3)} & K_{22}^{(3)} & K_{23}^{(3)} & K_{24}^{(3)} \\ K_{31}^{(3)} & K_{32}^{(3)} & K_{33}^{(3)} & K_{34}^{(3)} \\ K_{41}^{(3)} & K_{42}^{(3)} & K_{43}^{(3)} & K_{44}^{(3)} \end{array}\right] \end{array} \begin{array}{c} \\ 7 \\ 8 \\ 1 \\ 2 \end{array}$$



$$\mathbf{S} = \begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \left[\begin{array}{cc} K_{33}^{(1)} + K_{11}^{(2)} + K_{33}^{(3)} & K_{34}^{(1)} + K_{12}^{(2)} + K_{34}^{(3)} \\ K_{43}^{(1)} + K_{21}^{(2)} + K_{43}^{(3)} & K_{44}^{(1)} + K_{22}^{(2)} + K_{44}^{(3)} \end{array}\right] \end{array} \begin{array}{c} \\ 1 \\ 2 \end{array}$$

- Code numbers of concern during assembly of **<u>stiffness matrices</u>**: 1 ~ *NDOF*
- Now that both $\mathbf{P}$ and $\mathbf{S}$ are assembled, displacements $\mathbf{d}$ can be solved:

$$\boxed{\mathbf{P} = \mathbf{S}\mathbf{d}}$$

- Obtain member end displacements in the global coordinates, $\mathbf{v}$, from the joint displacements, $\mathbf{d}$, using code numbers
- Member-end displacements in local coordinates:

$$\mathbf{u} = \mathbf{T}\mathbf{v}$$

- Member-end forces in local coordinates (axial force = $Q_1 = -Q_3$):

$$\mathbf{Q} = \mathbf{k}\mathbf{u}$$

- Member end forces in global coordinates:

$$\mathbf{F} = \mathbf{T}^{\mathrm{T}}\mathbf{Q}$$

- Code numbers of concern during assembly of **<u>support reaction vector</u>**:

  $NDOF+1$ through $2(NJ)$

- Alternatively:

$$\mathbf{F} = \mathbf{K}\mathbf{v}$$
$$\mathbf{Q} = \mathbf{T}\mathbf{F}$$

$$\mathbf{F}_1 = \left[\begin{array}{c} F_1^{(1)} \\ F_2^{(1)} \\ F_3^{(1)} \\ F_4^{(1)} \end{array}\right] \begin{array}{c} 3 \\ 4 \\ 1 \\ 2 \end{array}$$

$$\mathbf{R} = \left[\begin{array}{c} R_3 \\ R_4 \\ R_5 \\ R_6 \\ R_7 \\ R_8 \end{array}\right] = \left[\begin{array}{c} F_1^{(1)} \\ F_2^{(1)} \\ F_3^{(2)} \\ F_4^{(2)} \\ F_1^{(3)} \\ F_2^{(3)} \end{array}\right] \begin{array}{c} 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array} \qquad \mathbf{F}_2 = \left[\begin{array}{c} F_1^{(2)} \\ F_2^{(2)} \\ F_3^{(2)} \\ F_4^{(2)} \end{array}\right] \begin{array}{c} 1 \\ 2 \\ 5 \\ 6 \end{array}$$

$$\mathbf{F}_3 = \left[\begin{array}{c} F_1^{(3)} \\ F_2^{(3)} \\ F_3^{(3)} \\ F_4^{(3)} \end{array}\right] \begin{array}{c} 7 \\ 8 \\ 1 \\ 2 \end{array}$$

(d) Assembly of Support Reaction Vector $\mathbf{R}$

- **Please review Kassimali 3.8** – a great summary of the complete procedure.

3.8 PROCEDURE FOR ANALYSIS

Based on the discussion presented in the previous sections, the following step-by-step procedure can be developed for the analysis of plane trusses subjected to joint loads.

1. Prepare an analytical model of the truss as follows.
   a. Draw a line diagram of the structure, on which each joint and member is identified by a number.

……

- One place we will implement differently from the textbook:

5c. Determine the member stiffness matrix in the local coordinate system, **k**, using Eq. (3.27); then calculate member end forces in the local coordinate system by using the stiffness relationship $\mathbf{Q} = \mathbf{k}\mathbf{u}$ (Eq. (3.7)). The member axial force, Q_a, equals the first element, Q_1, of the vector **Q** (i.e., $Q_a = Q_1$); a positive value of Q_a indicates that the axial force is compressive, and a negative value indicates that the axial force is tensile.

Set $Q_a = Q_3$, so that tension is positive