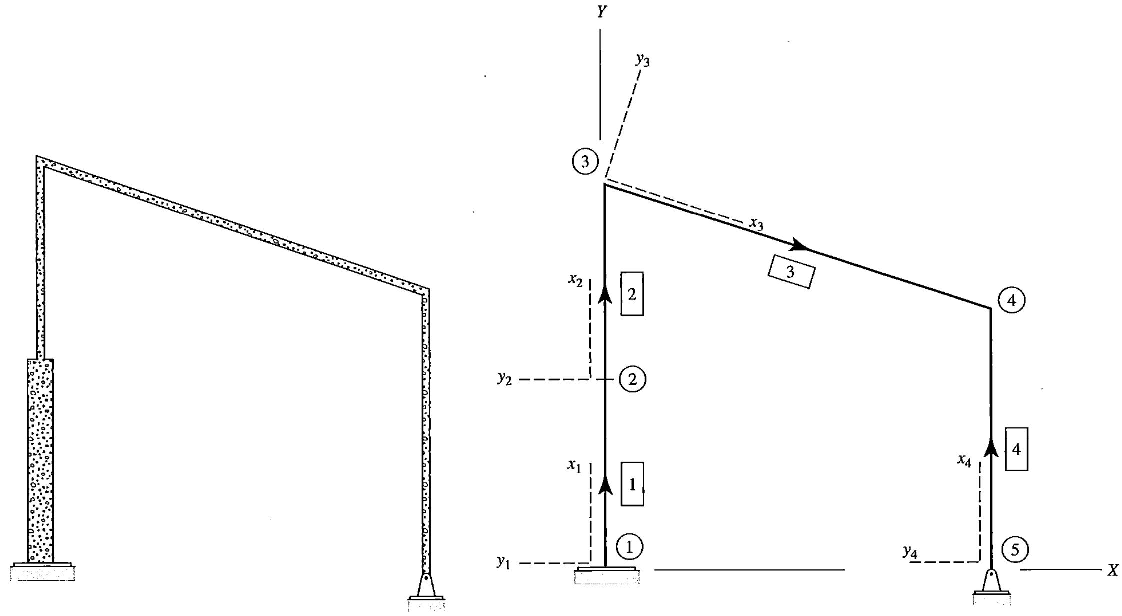
Chapter 5. Matrix Analysis of Plane Frames
5.1. Analytical Model (Kassimali §6.1)

- A plane frame is divided into members and joints so that: (a) all members are straight and prismatic; (b) all external reactions act only at the joints.



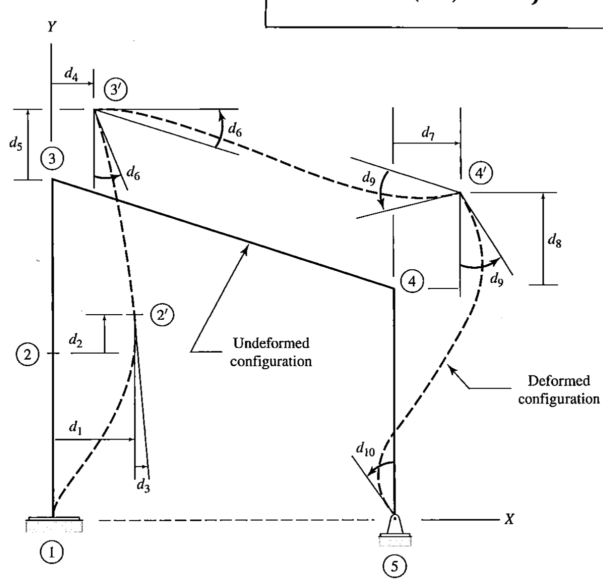
(a) Actual Frame

(b) Analytical Model Showing Global and Local Coordinate Systems

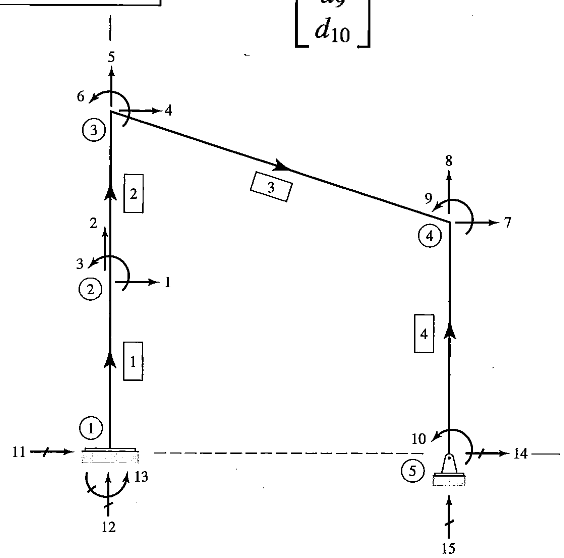
- A free joint has three DOFs, two translations and one rotation.
- Three coordinates (free and/or restrained) need to be defined at each joint: $NCJT$ = 3
- Joint displacement vector $\mathbf{d}$ is $NDOF \times 1$.
- Restrain coordinates: $NDOF + 1$ through $3(NJ)$.

$$\left.\begin{array}{l} NCJT = 3 \\ NDOF = 3(NJ) - NR \end{array}\right\} \text{ for plane frames}$$

$$\mathbf{d} = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_9 \\ d_{10} \end{bmatrix}$$



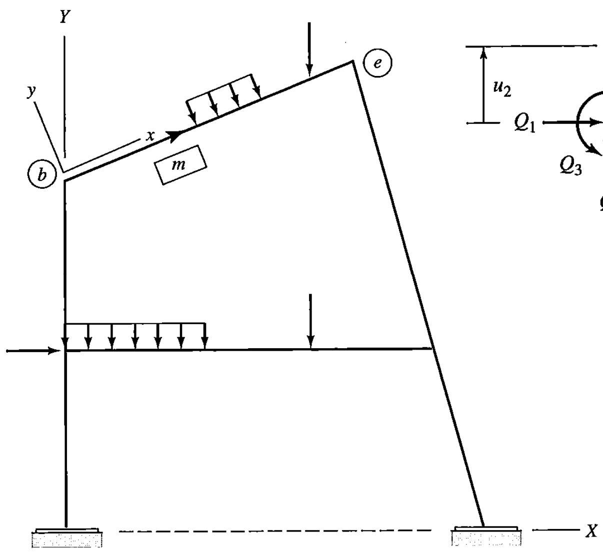
(c) Degrees of Freedom

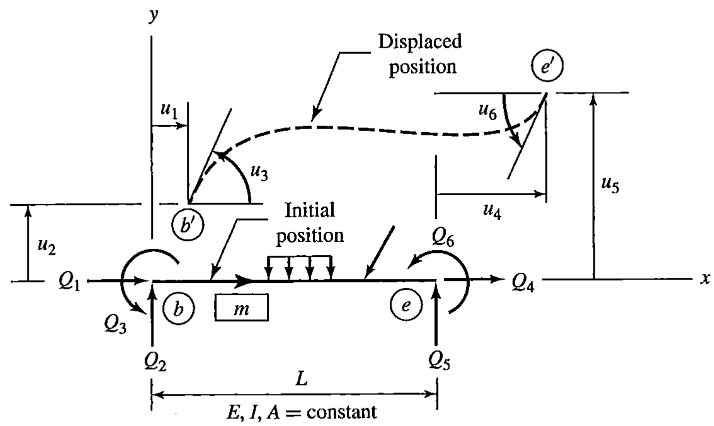


(d) Structure Coordinate Numbers ($NDOF = 10, NR = 5$)

5.2. Member Stiffness Relations in Local Coordinates

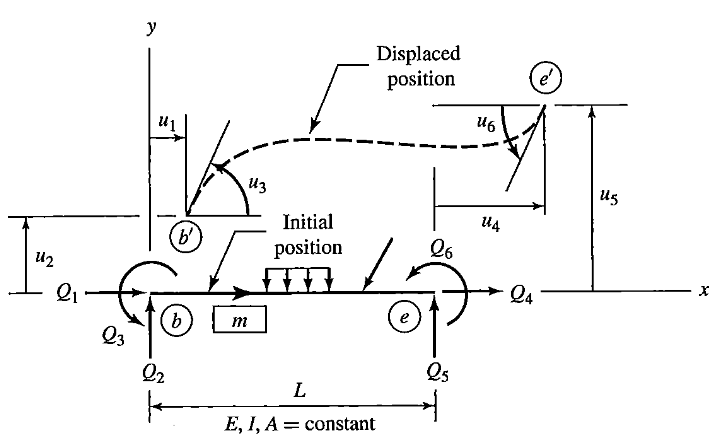
(Kassimali §6.2)



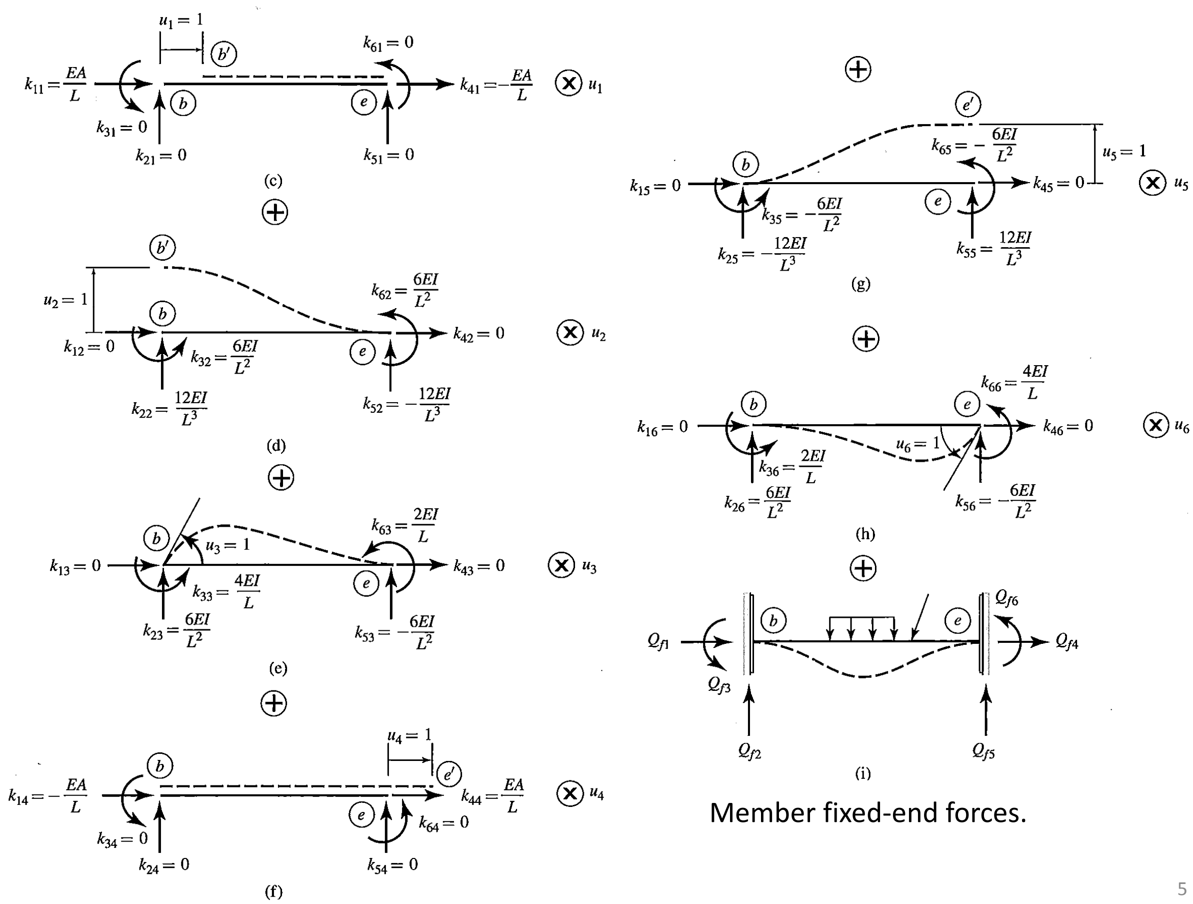


(b) Member Forces and Displacements
in the Local Coordinate System

- Six member end displacements (rotations): $u_1 \sim u_6$
- Six member end forces (moments): $Q_1 \sim Q_6$
- Translations/forces in the positive directions of the local x and y axes are positive
- Counterclockwise rotations/moments are positive
- A member's local end displacements and end forces are numbered by beginning at its end *b*, with the translation/force in the x direction numbered first, followed by the translation/ force in the y direction, and then rotation/moment.
- The displacements and end forces at the opposite end *e* are then numbered in the same sequential order.



(b) Member Forces and Displacements
in the Local Coordinate System



Member fixed-end forces.

$$Q_i = \sum_{j=1}^{6} \left(k_{ij} u_j\right) + Q_{fi} \qquad i = 1, 2, \ldots, 6$$

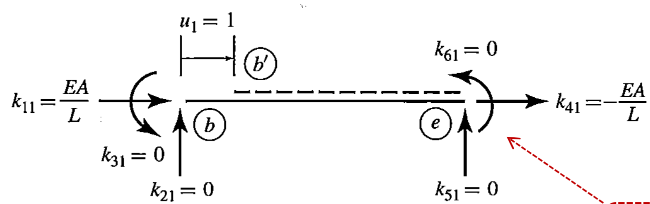
the stiffness coefficient k_{ij} represents the force corresponding to Q_i due to a unit value of the displacement u_j, and Q_{fi} denotes the fixed-end force corresponding to Q_i due to the external loads acting on the member.

$$\begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ Q_5 \\ Q_6 \end{bmatrix} = \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} & k_{15} & k_{16} \\ k_{21} & k_{22} & k_{23} & k_{24} & k_{25} & k_{26} \\ k_{31} & k_{32} & k_{33} & k_{34} & k_{35} & k_{36} \\ k_{41} & k_{42} & k_{43} & k_{44} & k_{45} & k_{46} \\ k_{51} & k_{52} & k_{53} & k_{54} & k_{55} & k_{56} \\ k_{61} & k_{62} & k_{63} & k_{64} & k_{65} & k_{66} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{bmatrix} + \begin{bmatrix} Q_{f1} \\ Q_{f2} \\ Q_{f3} \\ Q_{f4} \\ Q_{f5} \\ Q_{f6} \end{bmatrix}$$

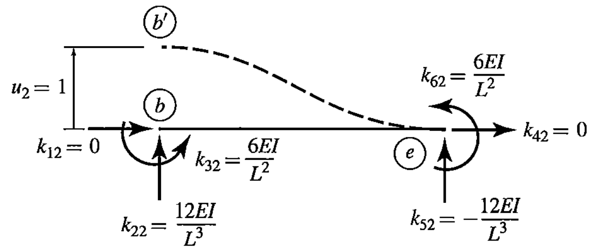
$$\mathbf{Q} = \mathbf{k}\mathbf{u} + \mathbf{Q}_f$$

6x1 member fixed-end force vector

- Member local stiffness matrix $\mathbf{k}$



$$\mathbf{k} = \frac{EI}{L^3}\begin{bmatrix} \frac{AL^2}{I} & 0 & 0 & -\frac{AL^2}{I} & 0 & 0 \\ 0 & 12 & 6L & 0 & -12 & 6L \\ 0 & 6L & 4L^2 & 0 & -6L & 2L^2 \\ -\frac{AL^2}{I} & 0 & 0 & \frac{AL^2}{I} & 0 & 0 \\ 0 & -12 & -6L & 0 & 12 & -6L \\ 0 & 6L & 2L^2 & 0 & -6L & 4L^2 \end{bmatrix}$$

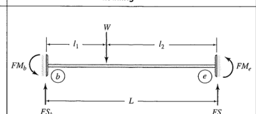
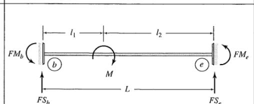
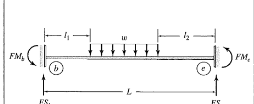


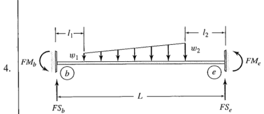
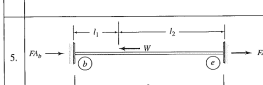
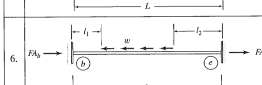
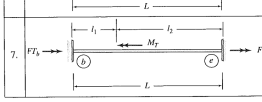
$$\mathbf{Q} = \mathbf{ku} + \mathbf{Q}_f$$

- Similar to member end forces in local coordinate system, $\mathbf{Q}$, a member's local fixed-end forces, $\mathbf{Q_f}$, are positive when in the positive directions of the local x and y axes, and the local fixed-end moments are considered positive when counterclockwise.

FIXED-END FORCE EXPRESSIONS

Fixed-End Moments, Shears and Axial Forces for Various Loading Conditions

No.	Loading	Equations for Fixed-End Moments, Shears, and Axial Forces
1.		$FS_b = \frac{Wl_2^2}{L^3}(3l_1 + l_2)$ $FM_b = \frac{Wl_1l_2^2}{L^2}$ $FS_e = \frac{Wl_1^2}{L^3}(l_1 + 3l_2)$ $FM_e = -\frac{Wl_1^2l_2}{L^2}$
2.		$FS_b = -\frac{6Ml_1l_2}{L^3}$ $FM_b = \frac{Ml_2}{L^2}(l_2 - 2l_1)$ $FS_e = \frac{6Ml_1l_2}{L^3}$ $FM_e = \frac{Ml_1}{L^2}(l_1 - 2l_2)$
3.		$FS_b = \frac{wL}{2}\left[1 - \frac{l_1}{L^4}(2L^3 - 2l_1^2L + l_1^3) - \frac{l_2^3}{L^4}(2L - l_2)\right]$ $FM_b = \frac{wL^2}{12}\left[1 - \frac{l_1^2}{L^4}(6L^2 - 8l_1L + 3l_1^2) - \frac{l_2^3}{L^4}(4L - 3l_2)\right]$ $FS_e = \frac{wL}{2}\left[1 - \frac{l_1^3}{L^4}(2L - l_1) - \frac{l_2}{L^4}(2L^3 - 2l_2^2L + l_2^3)\right]$ $FM_e = -\frac{wL^2}{12}\left[1 - \frac{l_1^3}{L^4}(4L - 3l_1) - \frac{l_2^2}{L^4}(6L^2 - 8l_2L + 3l_2^2)\right]$

No.	Loading	Equations for Fixed-End Moments, Shears, and Axial Forces
4.		$FS_b = \frac{w_1(L - l_1)^3}{20L^3}\left\{(7L + 8l_1) - \frac{l_2(3L + 2l_1)}{(L - l_1)} \times \left[1 + \frac{l_2}{L - l_1} + \frac{l_2^2}{(L - l_1)^2}\right] + \frac{2l_2^4}{(L - l_1)^3}\right\} + \frac{w_2(L - l_1)^3}{20L^3}\left\{(3L + 2l_1)\left[1 + \frac{l_2}{L - l_1} + \frac{l_2^2}{(L - l_1)^2}\right] - \frac{l_2^3}{(L - l_1)^2}\left[2 + \frac{15L - 8l_2}{L - l_1}\right]\right\}$ $FM_b = \frac{w_1(L - l_1)^3}{60L^2}\left\{3(L + 4l_1) - \frac{l_2(2L + 3l_1)}{L - l_1} \times \left[1 + \frac{l_2}{L - l_1} + \frac{l_2^2}{(L - l_1)^2}\right] + \frac{3l_2^4}{(L - l_1)^3}\right\} + \frac{w_2(L - l_1)^3}{60L^2}\left\{(2L + 3l_1)\left[1 + \frac{l_2}{L - l_1} + \frac{l_2^2}{(L - l_1)^2}\right] - \frac{3l_2^3}{(L - l_1)^2}\left[1 + \frac{5L - 4l_2}{L - l_1}\right]\right\}$ $FS_e = \left(\frac{w_1 + w_2}{2}\right)(L - l_1 - l_2) - FS_b$ $FM_e = \frac{L - l_1 - l_2}{6}[w_1(-2L + 2l_1 - l_2) - w_2(L - l_1 + 2l_2)] + FS_b(L) - FM_b$
5.		$FA_b = \frac{Wl_2}{L}$ $FA_e = \frac{Wl_1}{L}$
6.		$FA_b = \frac{w}{2L}(L - l_1 - l_2)(L - l_1 + l_2)$ $FA_e = \frac{w}{2L}(L - l_1 - l_2)(L + l_1 - l_2)$
7.		$FT_b = \frac{M_Tl_2}{L}$ $FT_e = \frac{M_Tl_1}{L}$