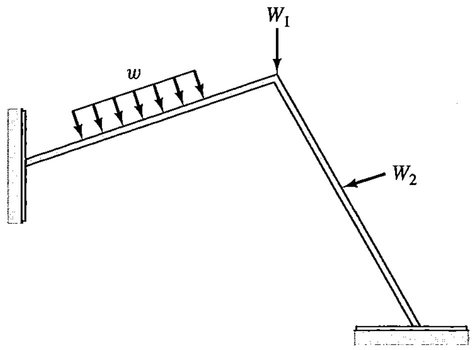
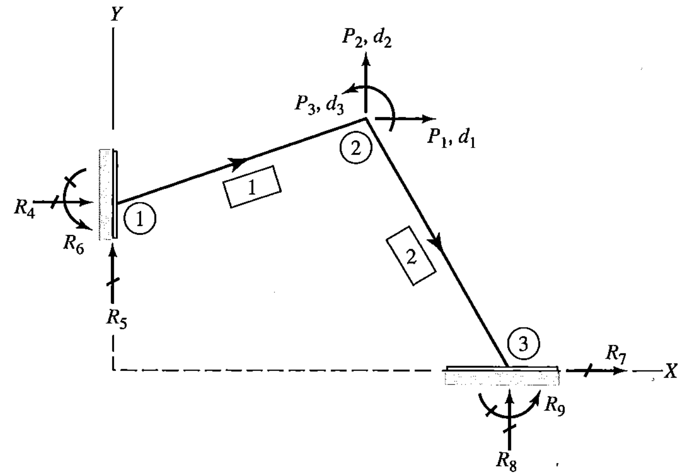
5.5. Structure Stiffness Relations (Kassimali §6.5)



(a) Plane Frame



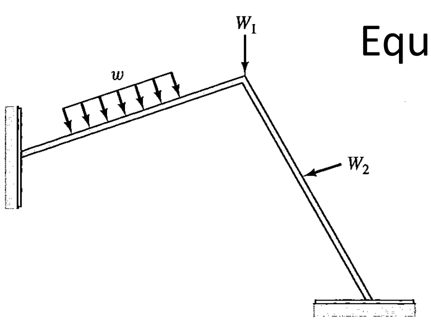
(b) Analytical Model

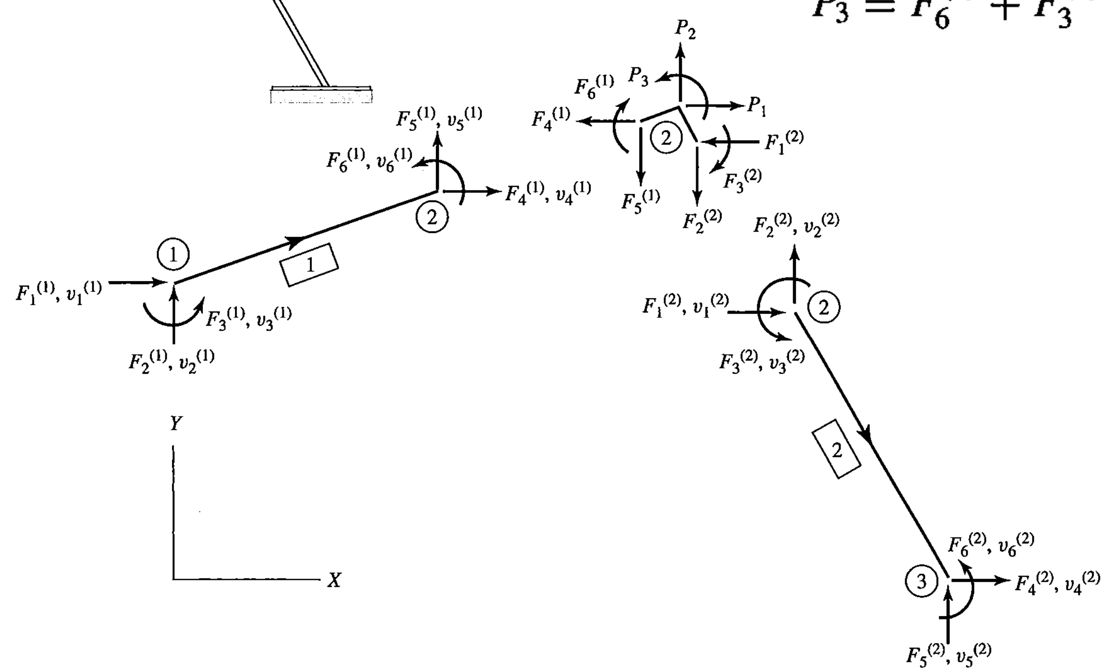
Equilibrium Equations

$$P_1 = F_4^{(1)} + F_1^{(2)}$$

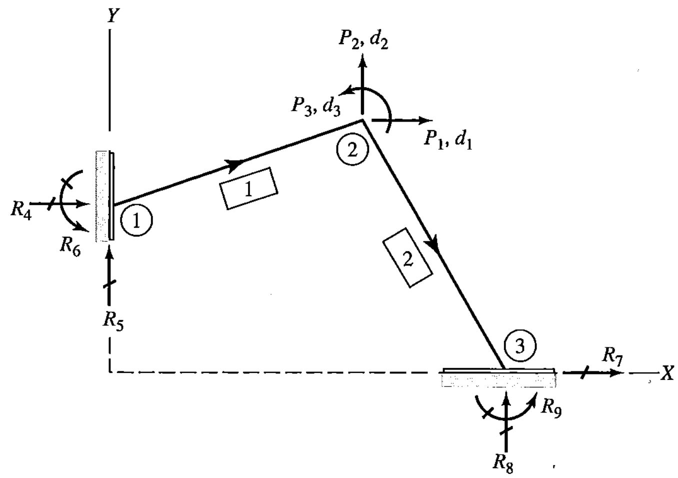
$$P_2 = F_5^{(1)} + F_2^{(2)}$$

$$P_3 = F_6^{(1)} + F_3^{(2)}$$





Compatibility Equations



$$v_1^{(1)} = v_2^{(1)} = v_3^{(1)} = 0 \qquad v_4^{(1)} = d_1 \qquad v_5^{(1)} = d_2 \qquad v_6^{(1)} = d_3$$

$$v_1^{(2)} = d_1 \qquad v_2^{(2)} = d_2 \qquad v_3^{(2)} = d_3 \qquad v_4^{(2)} = v_5^{(2)} = v_6^{(2)} = 0$$

Equilibrium Equations Compatibility Equations

$$P_1 = F_4^{(1)} + F_1^{(2)}$$
$$P_2 = F_5^{(1)} + F_2^{(2)}$$
$$P_3 = F_6^{(1)} + F_3^{(2)}$$

$$v_1^{(1)} = v_2^{(1)} = v_3^{(1)} = 0 \qquad v_4^{(1)} = d_1 \qquad v_5^{(1)} = d_2 \qquad v_6^{(1)} = d_3$$

$$v_1^{(2)} = d_1 \qquad v_2^{(2)} = d_2 \qquad v_3^{(2)} = d_3 \qquad v_4^{(2)} = v_5^{(2)} = v_6^{(2)} = 0$$

$$F_4^{(1)} = K_{41}^{(1)} v_1^{(1)} + K_{42}^{(1)} v_2^{(1)} + K_{43}^{(1)} v_3^{(1)} + K_{44}^{(1)} v_4^{(1)} + K_{45}^{(1)} v_5^{(1)} + K_{46}^{(1)} v_6^{(1)} + F_{f4}^{(1)} = K_{44}^{(1)} d_1 + K_{45}^{(1)} d_2 + K_{46}^{(1)} d_3 + F_{f4}^{(1)}$$

$$F_5^{(1)} = K_{51}^{(1)} v_1^{(1)} + K_{52}^{(1)} v_2^{(1)} + K_{53}^{(1)} v_3^{(1)} + K_{54}^{(1)} v_4^{(1)} + K_{55}^{(1)} v_5^{(1)} + K_{56}^{(1)} v_6^{(1)} + F_{f5}^{(1)} = K_{54}^{(1)} d_1 + K_{55}^{(1)} d_2 + K_{56}^{(1)} d_3 + F_{f5}^{(1)}$$

$$F_6^{(1)} = K_{61}^{(1)} v_1^{(1)} + K_{62}^{(1)} v_2^{(1)} + K_{63}^{(1)} v_3^{(1)} + K_{64}^{(1)} v_4^{(1)} + K_{65}^{(1)} v_5^{(1)} + K_{66}^{(1)} v_6^{(1)} + F_{f6}^{(1)} = K_{64}^{(1)} d_1 + K_{65}^{(1)} d_2 + K_{66}^{(1)} d_3 + F_{f6}^{(1)}$$

$$P_1 = \left(K_{44}^{(1)} + K_{11}^{(2)}\right) d_1 + \left(K_{45}^{(1)} + K_{12}^{(2)}\right) d_2 + \left(K_{46}^{(1)} + K_{13}^{(2)}\right) d_3 + \left(F_{f4}^{(1)} + F_{f1}^{(2)}\right)$$

$$P_2 = \left(K_{54}^{(1)} + K_{21}^{(2)}\right) d_1 + \left(K_{55}^{(1)} + K_{22}^{(2)}\right) d_2 + \left(K_{56}^{(1)} + K_{23}^{(2)}\right) d_3 + \left(F_{f5}^{(1)} + F_{f2}^{(2)}\right)$$

$$P_3 = \left(K_{64}^{(1)} + K_{31}^{(2)}\right) d_1 + \left(K_{65}^{(1)} + K_{32}^{(2)}\right) d_2 + \left(K_{66}^{(1)} + K_{33}^{(2)}\right) d_3 + \left(F_{f6}^{(1)} + F_{f3}^{(2)}\right)$$

$$P_1 = \left(K_{44}^{(1)} + K_{11}^{(2)}\right) d_1 + \left(K_{45}^{(1)} + K_{12}^{(2)}\right) d_2 + \left(K_{46}^{(1)} + K_{13}^{(2)}\right) d_3 + \left(F_{f4}^{(1)} + F_{f1}^{(2)}\right)$$

$$P_2 = \left(K_{54}^{(1)} + K_{21}^{(2)}\right) d_1 + \left(K_{55}^{(1)} + K_{22}^{(2)}\right) d_2 + \left(K_{56}^{(1)} + K_{23}^{(2)}\right) d_3 + \left(F_{f5}^{(1)} + F_{f2}^{(2)}\right)$$

$$P_3 = \left(K_{64}^{(1)} + K_{31}^{(2)}\right) d_1 + \left(K_{65}^{(1)} + K_{32}^{(2)}\right) d_2 + \left(K_{66}^{(1)} + K_{33}^{(2)}\right) d_3 + \left(F_{f6}^{(1)} + F_{f3}^{(2)}\right)$$

$$\mathbf{P} = \mathbf{S}\mathbf{d} + \mathbf{P}_f$$

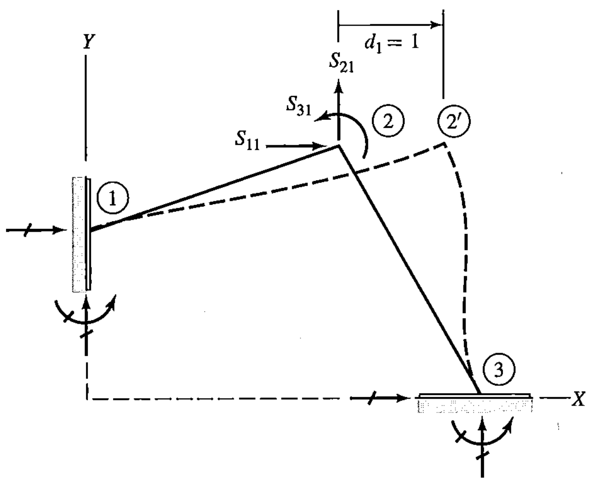
or

$$\boxed{\mathbf{P} - \mathbf{P}_f = \mathbf{S}\mathbf{d}}$$

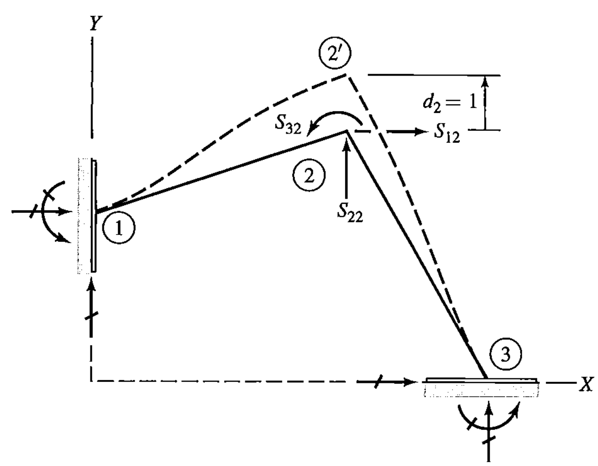
in which **S** represents the *NDOF* × *NDOF* structure stiffness matrix, and $\mathbf{P}_f$ is the *NDOF* × 1 structure fixed-joint force vector, for the plane frame with

$$\mathbf{S} = \begin{bmatrix} K_{44}^{(1)} + K_{11}^{(2)} & K_{45}^{(1)} + K_{12}^{(2)} & K_{46}^{(1)} + K_{13}^{(2)} \\ K_{54}^{(1)} + K_{21}^{(2)} & K_{55}^{(1)} + K_{22}^{(2)} & K_{56}^{(1)} + K_{23}^{(2)} \\ K_{64}^{(1)} + K_{31}^{(2)} & K_{65}^{(1)} + K_{32}^{(2)} & K_{66}^{(1)} + K_{33}^{(2)} \end{bmatrix} \qquad \textbf{(6.43)} \qquad \mathbf{P}_f = \begin{bmatrix} F_{f4}^{(1)} + F_{f1}^{(2)} \\ F_{f5}^{(1)} + F_{f2}^{(2)} \\ F_{f6}^{(1)} + F_{f3}^{(2)} \end{bmatrix}$$

an element S_{ij} of the structure stiffness matrix **S** represents the force at the location and in the direction of P_i required, along with other joint forces, to cause a unit value of the displacement d_j, while all other joint displacements are 0, and the frame is subjected to no external loads.



(a) First Column of **S** ($d_1 = 1, d_2 = d_3 = 0$)



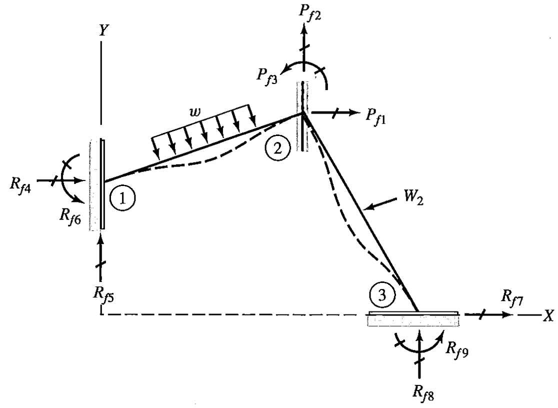
(b) Second Column of **S** ($d_2 = 1, d_1 = d_3 = 0$)

$$\mathbf{S} = \begin{bmatrix} K_{44}^{(1)} + K_{11}^{(2)} & K_{45}^{(1)} + K_{12}^{(2)} & K_{46}^{(1)} + K_{13}^{(2)} \\ K_{54}^{(1)} + K_{21}^{(2)} & K_{55}^{(1)} + K_{22}^{(2)} & K_{56}^{(1)} + K_{23}^{(2)} \\ K_{64}^{(1)} + K_{31}^{(2)} & K_{65}^{(1)} + K_{32}^{(2)} & K_{66}^{(1)} + K_{33}^{(2)} \end{bmatrix}$$

The structure stiffness matrix of a linear elastic structure must be symmetric.

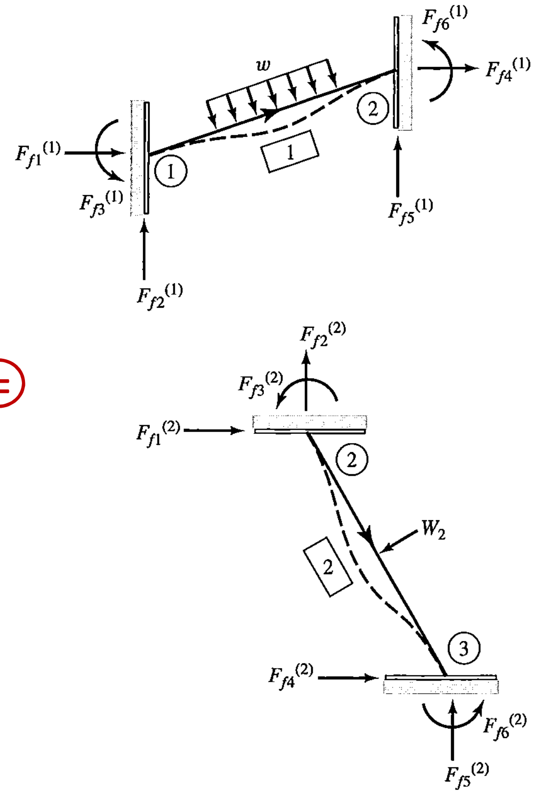
Structure fixed-joint force vector

$$\mathbf{P}_f = \begin{bmatrix} F_{f4}^{(1)} + F_{f1}^{(2)} \\ F_{f5}^{(1)} + F_{f2}^{(2)} \\ F_{f6}^{(1)} + F_{f3}^{(2)} \end{bmatrix}$$



(a) Fixed Frame Subjected to Member Loads

=

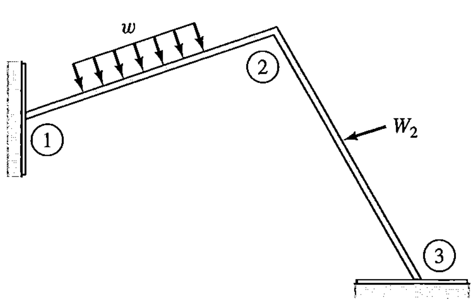


Equivalent Joint Load vector $\mathbf{P}_e = -\mathbf{P}_f$: another interpretation to structure fixed-joint force vector $\mathbf{P}_f$.

Consider the scenario containing only member loads, and without external joint loads, i.e. $\mathbf{P} = \mathbf{0}$,

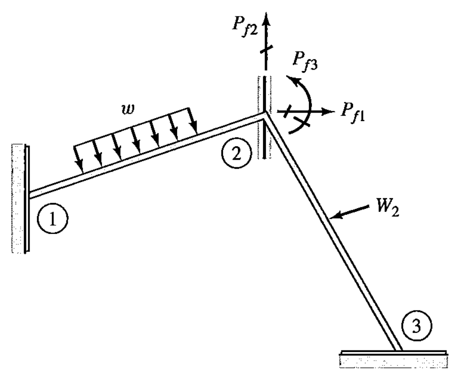
$$-\mathbf{P}_f = \mathbf{Sd}$$

The negatives of the structure fixed-joint forces cause the same joint displacements as the actual member loads.



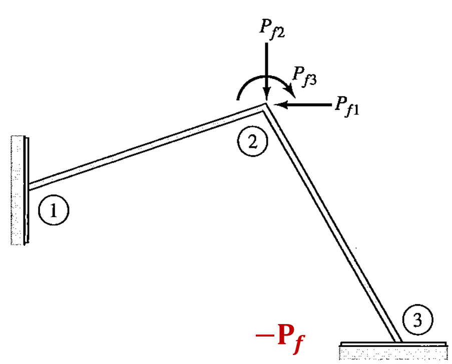
(a) Actual Frame Subjected to Member Loads

=



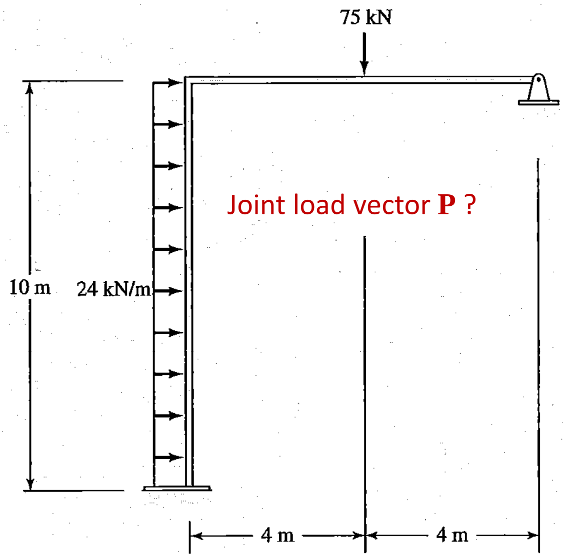
(b) Fixed Frame Subjected to Member Loads

+



(c) Actual Frame Subjected to Equivalent Joint Loads

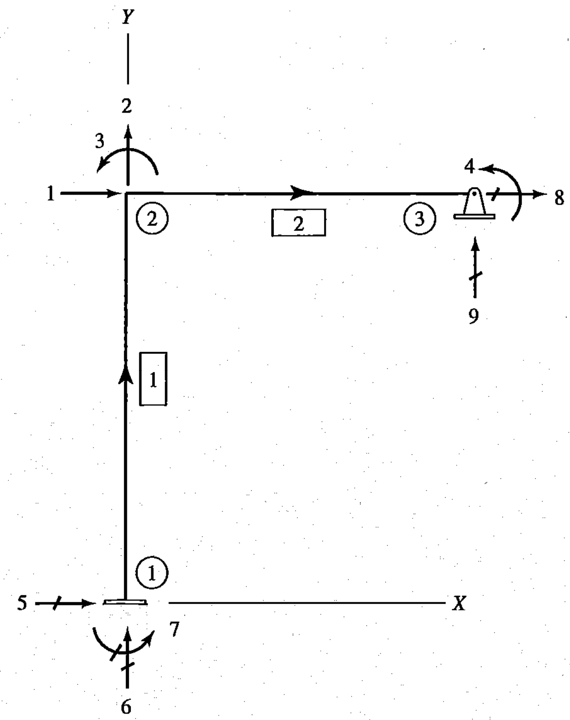
EXAMPLE 6.5



Joint load vector $\mathbf{P}$?

$E, A, I =$ constant
$E = 200$ GPa
$A = 4{,}740$ mm^2
$I = 22.2(10^6)$ mm^4

(a) Frame



(b) Analytical Model

Assembly can be performed using code numbers:

$$\boxed{\mathbf{P} - \mathbf{P}_f = \mathbf{S}\mathbf{d}}$$

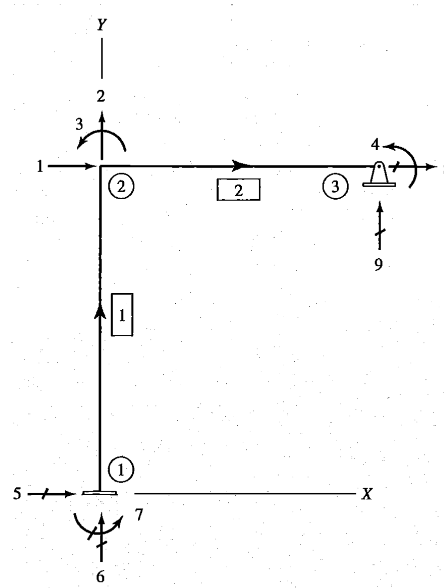
Assemble structures stiffness matrix **S**:

Member 1

$$\mathbf{K}_1 = \begin{array}{c} \begin{array}{cccccc} 5 & 6 & 7 & 1 & 2 & 3 \end{array} \\ \left[\begin{array}{ccc|ccc} 53.28 & 0 & -266.4 & -53.28 & 0 & -266.4 \\ 0 & 94{,}800 & 0 & 0 & -94{,}800 & 0 \\ -266.4 & 0 & 1{,}776 & 266.4 & 0 & 888 \\ \hline -53.28 & 0 & 266.4 & 53.28 & 0 & 266.4 \\ 0 & -94{,}800 & 0 & 0 & 94{,}800 & 0 \\ -266.4 & 0 & 888 & 266.4 & 0 & 1{,}776 \end{array}\right] \end{array} \begin{array}{c} 5 \\ 6 \\ 7 \\ 1 \\ 2 \\ 3 \end{array}$$

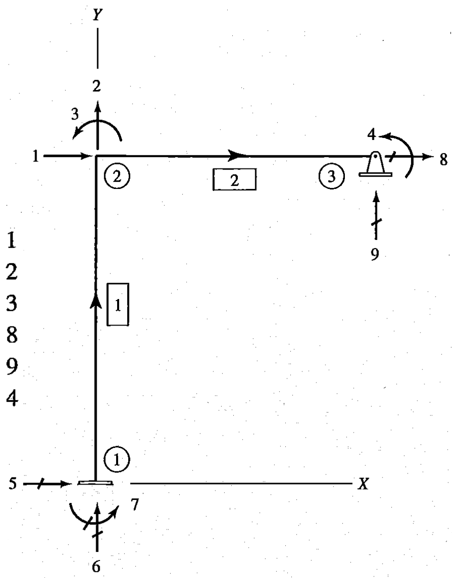
Contribution to structure stiffness matrix from Member 1:

$$\mathbf{S} = \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \left[\begin{array}{cccc} 53.28 & 0 & 266.4 & 0 \\ 0 & 94{,}800 & 0 & 0 \\ 266.4 & 0 & 1{,}776 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array}$$



(b) Analytical Model

Member 2



(b) Analytical Model

$$\mathbf{K}_2 = \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 8 & 9 & 4 \end{array} \\ \left[\begin{array}{ccc|cc|c} 118{,}500 & 0 & 0 & -118{,}500 & 0 & 0 \\ 0 & 104.06 & 416.25 & 0 & -104.06 & 416.25 \\ 0 & 416.25 & 2{,}220 & 0 & -416.25 & 1{,}110 \\ \hline -118{,}500 & 0 & 0 & 118{,}500 & 0 & 0 \\ 0 & -104.06 & -416.25 & 0 & 104.06 & -416.25 \\ \hline 0 & 416.25 & 1{,}110 & 0 & -416.25 & 2{,}220 \end{array}\right] \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 8 \\ 9 \\ 4 \end{array}$$

$$\mathbf{S} = \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 53.28 + 118{,}500 & 0 & 266.4 & 0 \\ 0 & 94{,}800 + 104.06 & 416.25 & 416.25 \\ 266.4 & 416.25 & 1{,}776 + 2{,}220 & 1{,}110 \\ 0 & 416.25 & 1{,}110 & 2{,}220 \end{bmatrix} \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}$$

$$= \begin{bmatrix} 118{,}553 & 0 & 266.4 & 0 \\ 0 & 94{,}904 & 416.25 & 416.25 \\ 266.4 & 416.25 & 3{,}996 & 1{,}110 \\ 0 & 416.25 & 1{,}110 & 2{,}220 \end{bmatrix}$$

To assemble fixed-joint force vector $\mathbf{P}_f$, start with fixed-end forces for each member:

$$\mathbf{F}_f = \mathbf{T}^T \mathbf{Q}_f$$

Member 1

$$\mathbf{F}_{f1} = \left[\begin{array}{c} -120 \\ 0 \\ 200 \\ \hline -120 \\ 0 \\ -200 \end{array}\right] \begin{array}{c} 5 \\ 6 \\ 7 \\ 1 \\ 2 \\ 3 \end{array}$$

Member 2

$$\mathbf{F}_{f2} = \left[\begin{array}{c} 0 \\ 37.5 \\ 75 \\ \hline 0 \\ 37.5 \\ \hline -75 \end{array}\right] \begin{array}{c} 1 \\ 2 \\ 3 \\ 8 \\ 9 \\ 4 \end{array}$$

$$\mathbf{P}_f = \begin{bmatrix} -120 \\ 37.5 \\ -200 + 75 \\ -75 \end{bmatrix} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} = \begin{bmatrix} -120 \\ 37.5 \\ -125 \\ -75 \end{bmatrix}$$

$\boxed{\mathbf{P} - \mathbf{P}_f = \mathbf{S}\mathbf{d}}$ ⟹ Displacements $\mathbf{d}$ ⟹ Member-end displacements (global): $\mathbf{v}$ ⟹ Member-end displacements (local): $\mathbf{u} = \mathbf{T}\mathbf{v}$

⟹ Member-end forces (local): $\mathbf{Q} = \mathbf{k}\mathbf{u} + \mathbf{Q}_f$ ⟹ Member-end forces (local): $\mathbf{F} = \mathbf{T}^T\mathbf{Q}$ ⟹ Support (restraint) reactions

Member-end forces (local): $\mathbf{Q} = \mathbf{k}\mathbf{u} + \mathbf{Q}_f$ ⟹ Shear/moment diagrams

Equivalent Joint Load vector $\mathbf{P}_e = -\mathbf{P}_f$ causes the same joint displacements as the actual member loads.

$$\mathbf{P}_e = -\mathbf{P}_f = \begin{bmatrix} 120 \\ -37.5 \\ 125 \\ 75 \end{bmatrix}$$

- Please review **Section 6.6** in Kassimali– a great summary of the complete procedures; and two more examples: **Example 6.6 and 6.7**.

6.6 PROCEDURE FOR ANALYSIS

Using the concepts discussed in the previous sections, we can now develop the following step-by-step procedure for the analysis of plane frames by the matrix stiffness method.

1. Prepare an analytical model of the structure, identifying its degrees of freedom and restrained coordinates (as discussed in Section 6.1). Recall that for horizontal members, the coordinate transformations can be avoided by selecting the left-end joint of the member as the beginning joint.

2. Evaluate the structure stiffness matrix $\mathbf{S}(NDOF \times NDOF)$ and fixed-joint force vector $\mathbf{P}_f\,(NDOF \times 1)$. For each member of the structure, perform the following operations:

   a. Calculate the length and direction cosines (i.e., $\cos\theta$ and $\sin\theta$) of the member (Eqs. (3.62)).