Chapter 3: Mean-field and Landau Theories.

Continuing our study of the Ising model, we will now like to ascend to $d=3$. As of the date of writing of these notes, the 3-d Ising model has not succumbed to an analytical solution. We, nevertheless, know a lot about the phase diagram of these systems. By using the argument of Peierls, we know that there will be an ordered phase at ~~high~~ low temperatures and a disordered phase at high temperatures. These phases are "separated" by a critical point (T_c). Much is known about this from series expansion, numerical work and physical considerations. We shall now use the Ising model in 3-D as a means of developing further ideas for the analysis of Interacting systems.

Feyman Variational Principle.

~~With~~ Suppose I have a Hilbert space $\{x\}$ and a Hamiltonian ~~of~~ $\mathcal{H}$ defined on ~~It~~ it. Suppose now that this problem is "too hard" (such as the Ising models in 3D) to solve, what do we do? One approach would be to "change the problem"! Do something that we *can* do! The idea then is to write down a trial Hamiltonian $H_0(\lambda)$ (which depends on some parameter λ), on the *same* Hilbert space. The remarkable point to be noted is that $H_0(\lambda)$ *need not* have the same symmetries as H!!! All of this sounds a bit strange, but let us see ~~what~~ a bit more.

Now the partition function of the real problem is

$$Z = \sum_x{}' e^{-\beta\mathcal{H}(x)}.$$

and that of the trail Hamiltonian as $H_0(\lambda)$ (we will now suppress the λ variable)

$$Z_0 = \sum_x{}' e^{-\beta\mathcal{H}_0(x)}$$

We now see that

$$Z = \sum_{x}{}' e^{-\beta(H_0(x) + H(x) - H_0(x))}$$

Since we in the classical world,

$$Z = \sum_{x}{}' e^{-\beta H_0(x)} e^{-\beta(\underbrace{H(x)-H_0(x)}_{W})}$$

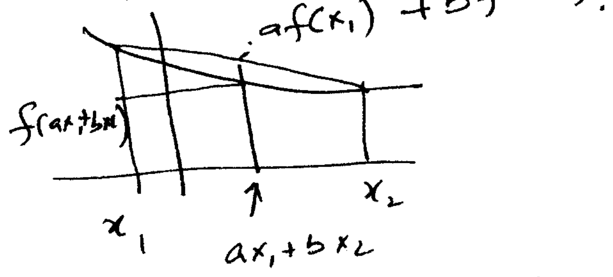
$$= Z_0 \left\langle e^{-\beta W} \right\rangle_0 .$$

Now we ~~not~~ make use of a mathematical identity called the Jensen inequality. For this we need a defin. A function $f(x)$ is said to be convex, if

$$f(ax_1 + bx_2) \leq a f(x_1) + b f(x_2)$$

$a+b=1$.

Pictorially ~~a~~ a convex function looks like this



Note that e^{-x} is a convex function!

Jensens ~~eqn~~ inequality state that, if there ~~the~~ and is a probability distribution $p(x)$, $f(x)$ is a convex function then $\int dx\, p(x) f(x) = \langle f(x) \rangle$

$$\int dx\, x\, p(x) = \langle x \rangle$$

and $\langle f(x) \rangle \geq f(\langle x \rangle)$.

A corrolaty for any function $f(x)$

$$\langle e^{f(x)} \rangle \geq e^{\langle f(x) \rangle}.$$

This follows from the convexity of exponential function.

Thus we see that

$$Z = Z_0 \langle e^{-\beta W} \rangle_0$$

$$\geq Z_0\, e^{-\beta \langle W \rangle_0}$$

$\Rightarrow$ $F = -T \ln Z$

$$F \leq F_0 + \langle H - H_0 \rangle_0 . \; !$$

We see that we can obtain a <u>bound</u> for the free energy from H_0.

Of what use is this? Note that $H_0(\lambda)$, i.e, there are a whole range of parameters in this Hamiltonian as λ stands for the quantities that the easy Hamiltonian H_0 depends on. We can now find the R.H.S which is

$$F_0(\lambda) + \langle H - H_0(\lambda) \rangle_0$$

and optimize (minimize) this quantity with respect to λ. This will get us the best trial Hamiltonian ~~the~~ of the <u>form</u> we have assumed. One can then obtain all the necessary information about observables by ~~seeing the~~ finding expectation values using H_0.

Two points:

→ The decision of what H_0 to use requires good physical understanding of the system

→ The parameters λ have some "physical interpretation", and this is best seen by an example.

Let us use the Feynmann principle to solve the d-cubic lattice Ising model.

$$H = -J \sum_{\langle ij \rangle}' S_i S_j \quad \text{at temperature } T.$$

Now, for $d > 2$ it has not been "solved" exactly analytically. What H_0 to choose? Well How about

$$H_0 = -B \sum_i' S_i$$

where B is some magnetic field. In this case $\lambda \equiv B$.

Now we can easily calculate

$$F_t(B) = F_0(B) + \langle H - H_0(B) \rangle_0$$

$$F_0(B) = -NT \ln 2\cosh(\beta B)$$

$$\langle H - H_0 \rangle_0 = - dNJ \tanh^2 \beta B + NB \tanh(\beta B)$$

$$f_t(B) = \frac{F_t(B)}{N} = -T \ln(2\cosh(\beta B)) - dJ \tanh^2 \beta B + B \tanh(\beta B)$$

$$\frac{\partial f_t(B)}{\partial B} = 0$$

$$-\tanh \beta B + (-2dJ \tanh \beta B + \beta B) \operatorname{sech}^2 \beta B + \tanh \beta B = 0$$

$$\Rightarrow \qquad B = 2dJ \tanh \beta B.$$

The solution of this equation is

$$B = 0 \qquad 2dJ\beta < 1$$

$$B \neq 0 \qquad 2dJ\beta \geq 1.$$

Let us call ~~β_c~~ $2dJ\beta_c = 1$. We also define $z = 2d$ is the coordination number.

For $T \lesssim T_c$, we can do some approximate analysis since B will be small.

$$\Rightarrow \qquad B = zJ\left(\beta B - \frac{(\beta B)^3}{3}\right)$$

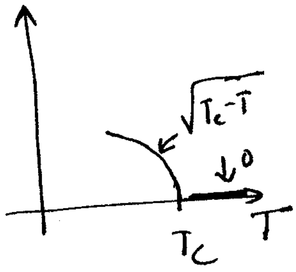
$$\sim \qquad 1 = zJ\left[\beta - \frac{\beta^3 B^2}{3}\right]$$

$$\Rightarrow \qquad zJ\beta_c = zJ\left(\beta - \frac{\beta^3 B^2}{3}\right)$$

$$\Rightarrow \qquad B = \sqrt{3\beta_c^{-3}(\beta - \beta_c)} \qquad \beta > \beta_c.$$

$$= \sqrt{3T_c(T_c - T)}$$

So B is nonzero below T_c and zero above.



Note a key point about this approach λ, here ~~[illegible]~~ B, is generically dependent on the conditions under which the system is kept. Here B depends on ~~#~~ temperature.

Now, for this example, what is T_c? Is it the transition temperature ? of the Ising model. Let us check by calculating the magnetization.

$$M = \tanh \beta B$$

For $T > T_c$ $M = 0$! and $T < T_c$ $M \neq 0$. Thus T_c is the transition temperature!.

T_c by our calculation $= zJ$!. A simple analytical result.

Sadly, this is wrong in $d=1$! But is quite ok in $d=2$, where we get $T_c = 4J$. This is ~~some~~ somewhat larger than the analytical value that we found but nevertheless quite ok. In $d=3$, where we have no analytic solution we find $T_c = 6J$. Note also that the analysis predicts that $M \sim \sqrt{3}\left(\frac{|T-T_c|}{T_c}\right)^{1/2} \theta(T_c - T)$ and this is independent of d!

Traditional Mean field Theory.

There is a more standard way of approching statmech (both classical and quantum problems) by a method call meanfied theory. Imagine that the Hamiltonian is written as a ~~prodca~~ sum of products of "statistical" or "dynamic" quantits

$$H = \sum_{\alpha}' \underbrace{A_\alpha B_\alpha}_{\text{product}}$$

The generic meanfield ansatz is the follow

$$A_\alpha B_\alpha \equiv \langle A_\alpha \rangle B_\alpha + \langle B_\alpha \rangle A_\alpha - \langle A_\alpha \rangle \langle B_\alpha \rangle .$$

What is $\langle A_\alpha \rangle$? This is the <u>average</u> of <u>mean</u> value of the "operator" $\langle A_\alpha \rangle$
The last subtraction is to prevent double counting.

The physical idea of mean-field can be understood by looking at the term $A_\alpha B_\alpha$. If we ~~see~~ think about this term, we can interpret this as a "fluctuating field" B_α acting on A_α. The idea of ~~the~~ "mean field" theory is replace this fluctuating field by a mean field $\langle B_\alpha \rangle$, such that A_α now sees a constant field. What is missed out?

$$A_\alpha B_\alpha = (\langle A_\alpha \rangle + \delta A_\alpha)(\langle B_\alpha \rangle + \delta A_\alpha)$$

$$\langle A_\alpha B_\alpha \rangle = \langle A_\alpha \rangle \langle B_\alpha \rangle + \langle \delta A_\alpha \delta B_\alpha \rangle$$

This is an exact result. Now

$$\langle A_\alpha B_\alpha \rangle_{MFT} = \langle A_\alpha \rangle \langle B_\alpha \rangle$$

We see that MFT misses out on fluctuations. Thus MFT is likely to give good qualitative results if fluctuations are *not* important. We will see later what general conditions need to be satisfied ~~to~~ so that this is true.

With this understanding we can now write the meanfield Hamiltonian as

$$H_{MF} = \sum_{\alpha}{}' \left(\langle A_\alpha \rangle B_\alpha + \langle B_\alpha \rangle A_\alpha - \langle A_\alpha \rangle \langle B_\alpha \rangle \right)$$

Now the key point is that we do not know $\langle A_\alpha \rangle$ and $\langle B_\alpha \rangle$!

Now by our argument

$$\langle A_\alpha \rangle = \mathrm{tr}(\rho A_\alpha)$$

where $\rho = \dfrac{e^{-\beta \mathcal{H}}}{Z}$!

But we cannot solve this! ~~Now the idea~~ ~~of Meanfield theory is~~ Now the second idea of MFT is to replace $\langle A_\alpha \rangle$ by $\langle A_\alpha \rangle_{H_{MFT}}$ i.e

$$\langle A_\alpha \rangle \simeq \langle A_\alpha \rangle_{H_{MFT}}$$

$$\langle A_\alpha \rangle_{H_{MFT}} = \mathrm{tr}(\rho_{MFT} A_\alpha)$$

$$\rho_{MFT} = \frac{e^{-\beta H_{MFT}}}{Z_{MFT}} \; !$$

We now see that we obtain as many equations as unknowns.

$$\langle A_\alpha \rangle = \mathrm{tr}(\rho_{MFT} A_\alpha)$$

$$\langle B_\alpha \rangle = \mathrm{tr}(\rho_{MFT} B_\alpha).$$

These (generically nonlinear) equations provide the self consistency condition of the MFT. Once these equations are solved, we obtain $\langle A_\alpha \rangle$ and $\langle B_\alpha \rangle$ and this allows for the calculation of any observable O, $\langle O \rangle = \mathrm{tr}(\rho_{MFT} O)$.
An Important point is that H_{MFT} is usually easy to solve.

Let us see ~~that~~ MFT in action. Back to our Ising model.

$$H = -J \sum_{\langle ij \rangle}' S_i S_j.$$

This is in precisely the form that we used during MFT in general.
Now define $\langle S_i \rangle = M$

Thus $S_i S_j \underset{\overset{\uparrow}{MFT}}{\simeq} M S_i + M S_j - M^2.$

$$H_{MFT} = -J \sum_i \sum_{j/i}' M S_i - dNM^2$$

$$H_{MFT} = -\sum_i{}' (zJM)\, S_i \qquad - dNM^2\,!$$

The self consistency eqn is

$$M = \mathrm{tr}(\rho_{MTT}\, S_i)$$

$$\Rightarrow \quad M = \tanh \beta zJM .$$

We see that

$$\begin{matrix} M \neq 0 \\ M = 0 \end{matrix} \left\{ \begin{matrix} \beta > \beta_c \\ \beta \leq \beta_c \end{matrix} \right.$$

~~and thus we~~ where $\beta_c = \dfrac{1}{zJ}$ or $T_c = zJ$!

We see that for $\beta > \beta_c$

$$M \sim \sqrt{T_c - T} \; !$$

Note that

① This is exactly same as the result from the Feynmann Variational principle. This is not an accident! These things are related (find out how!) Also we now understand the meaning of the ~~Feynmann~~ B field introduced in the H_0 used in the Feynmann calculation. B is the self-consistent "molecular field" generated by other spins on a given spin. In other words $B = zJM$. Note that such a molecular field is nonzero only in the broken symmetry phase! This is reflected in the symmetry of the $\mathcal{H}_{MFT}$ Hamiltonian.

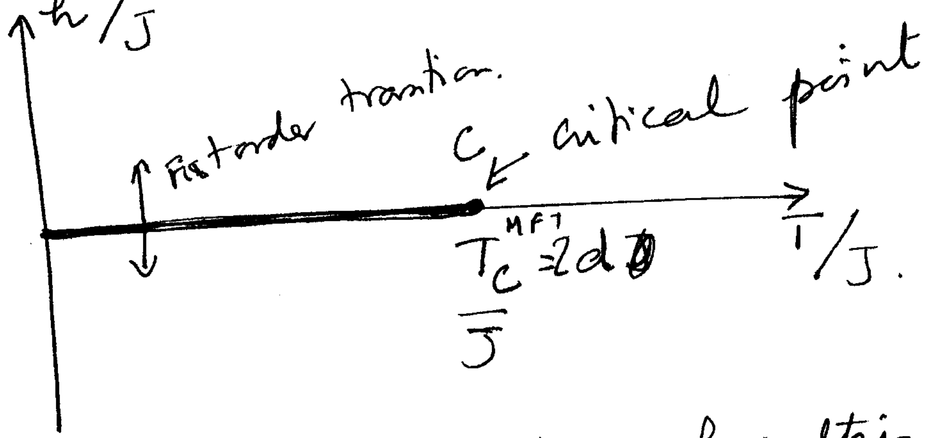
② The interpretation of $\langle\ \rangle$ in the above discussion has been based on classical stat mech. MFT can also be used in purely quantum problems (at zero temperature). For example, one can analyze the ground state of TFIM using mean field theory. (Try this out!)

~~We shall now see some further examples of MFT.~~

③ Mean field theory is non-perturbative; there is no small parameter assumed. ~~Theory~~ The key idea of MFT is the fluctuations are ignored. Another important part is that mean field self consistent equations are generically nonlinear and may require numerical solution.

Meanfield theory can be used to generate the full phase diagram and usually provides the first guide to getting a handle on the problem.

For example, For the Ising model, we The full phase digram is in the h-T plane



The meanfield ansatz for this problem is

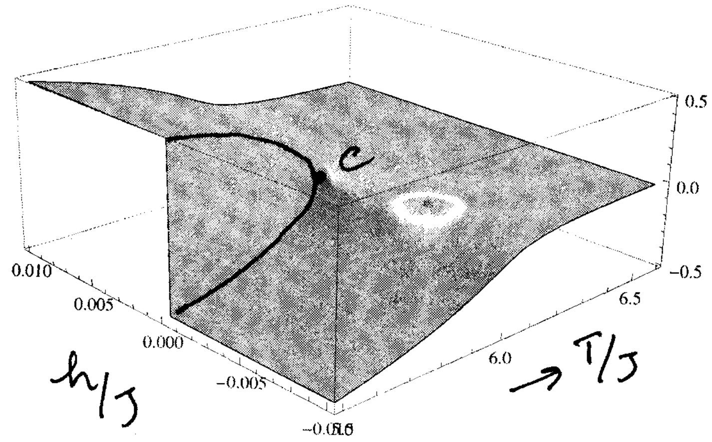
$$M = \tanh(\beta(JzM + h))$$

This has to solved for M for a given β and h, and one can make a plot of the resulting magnetization M.

We learn a few things from here

The solid line at $h=0$ going from $T=0$ to $T=T_c$ is a 1st order transition line.

If we traverse the phase digram at ~~h~~ at a fixed temperature below T_c, along a vertical line



We will concenter a 1st order line. as seen above. The line of first order transitions terminates at the critical point C which corresponds to $(T=T_c$ and $h=h_c=0)$. Note that this phase digram is very similar to the phase diagram of a standard material (such as CO_2) near the critical point. The first or line is like the liquid-gas phase boundary and this first ~~of~~ order line terminates at the critical point. As you have seen before this is not an accident. The liquid gas transition can be modeled by the Ising model, and in fact is the lattice gas model.

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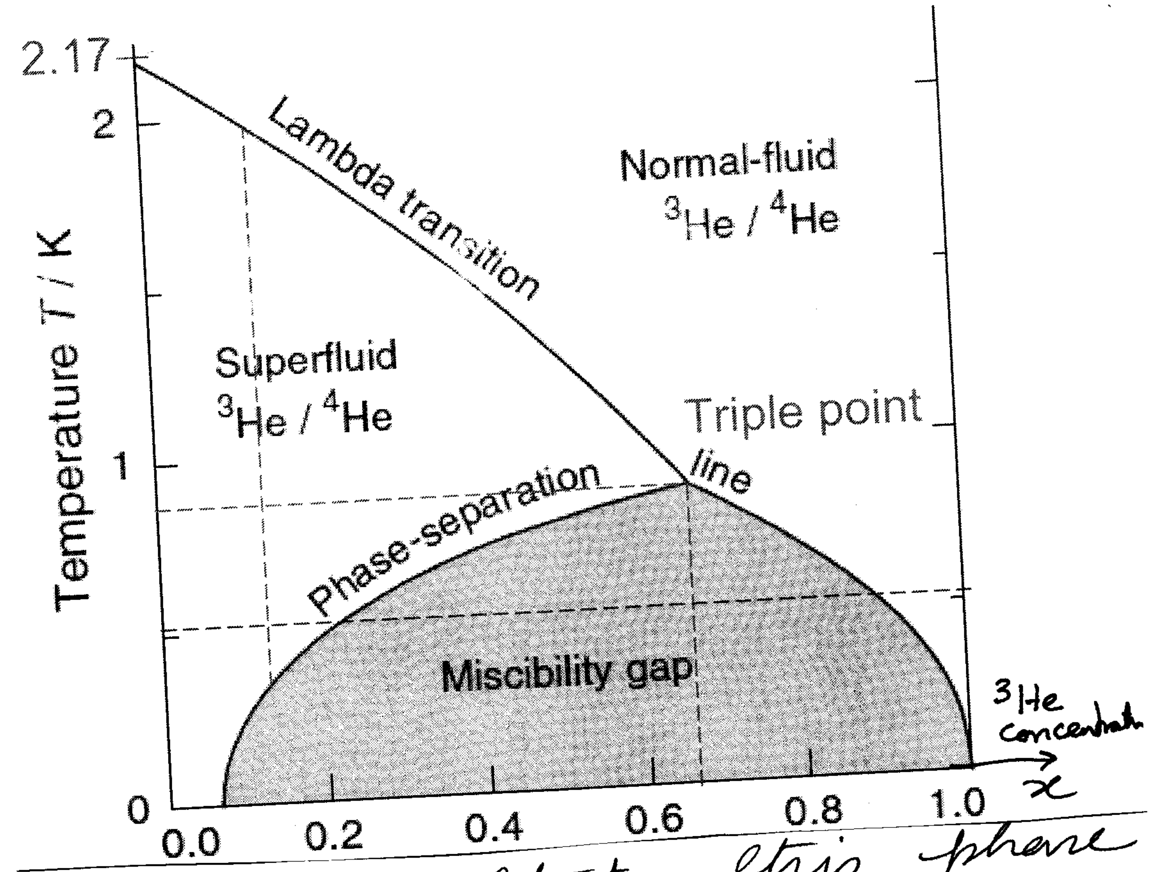
We will now see one more application of meanfield theory. To motivate the example, we will ask the following question. We have diffrent phases and phase digrams. Are all of them like the Ising model (~~First~~ 1st order line terminating at a critical point) or are there even other interesting things? The answer turns out to be an emphatic "yes"!! Let us explore.

Helium gas is well known to many experimental physicists who actually prefer Helium in the liquid form. (Actually lots of physicists prefer certain things in liquid form!) When we cool Helium at about 4K, it liquifies. (all this is at atmospheric pressure) Liquid helium ~~at~~ this temperature $\lesssim 4$ K is just like any other liquid although drinking is not recommended. Upon cooling this further something spectacular occurs. At about 2K, something Liquid helium suddenly turns into a new ~~like~~ kind of fluid that appears exceptionally placid, and shows exotic properties - called the superfluid

In fact there is a phase change from an ordinary liquid ~~to~~ a superfluid is ~~is~~ accompanied by a specific heat anomaly which looks like a λ - and hence the transition is called the λ-transition and The ~~trans~~ transition temperature is called T_λ. This transition taught us a lot of physics, infact has provided some crucial tests of key theories of physics. We will not have the time to go into all this, but suffice it here to know that the broken symmetry in the superfluid phase is a U(1) (continuous) symmetry (SO(2)), and the order parameter is a complex number. Now, with this background, we move on to the problem ~~of~~ at hand. What happens if we mix a bit of ^3He into ^4He? (Actually this is a very important thing from the point of view of experiment) This problem has been studied in great detail experimentally and the phase diagram of the system is as shown.



Now we see that this phase diagram is certainly more complex than the simple Ising model. It has a ~~set of~~ line of second order phase transitions, *and* a line of first order phase transitions.

We will make a model that has features similar to this phase diagram. Actually, we will steal ~~from~~ (shamelessly) from the work of Blume, Emery and Griffiths (PRA, 4, 1071 (1971), highly recommended reading). The remarkable point is that these authors took some bold steps to understand the phase diagram: They asked the question; If Ising model can describe the liquid gas transition, can one make an Ising like model for the phase diagram of ^{4}He-^{3}He mixtures. They did! The model is on a d-cubic lattice (d=3 for our system, but we can keep our discussion general).

The arena for the BEG model is the d-cubic lattice, N sites labelled by i.

The degrees of freedom are Ising spins, but <u>spin 1</u>, $S_{iz} \equiv \{1, 0, -1\}$.

What is the connection between spin 1 Ising variables and ~~the~~ He? Well, $S_i = \pm 1$ means that the site has a 4He atom, while $S_i = 0$ means that there is a 3He atom.

Hamiltonian of the system has a term that encourages superfluidity (or condensation)

$$H_J = -J \sum_{\langle ij \rangle}' S_i S_j$$

(This is like the usual Ising model).

Then there is the interactions between the atoms on different sites.

$$H_V = -V_{44}\sum_{(ij)}{}' S_i^2 S_j^2$$
$$- V_{33}\sum_{(ij)}{}' (1-S_i^2)(1-S_j^2)$$
$$- V_{34}\sum_{(ij)}{}' (1-S_i^2)S_j^2 + (1-S_j^2)S_i^2.$$

V_{44} is interaction between two ^{4}He ~~and~~ atom on nearest neighbor sites.

Since we need the number of particles in the system to controlled. We also need to have chemical potential that fix the number of particles

$$H_\mu = -\mu_3\left(\sum_i{}'(1-S_i^2)\right) - \mu_4\left(\sum_i{}' S_i^2\right).$$
$$= -(\mu_4-\mu_3)\sum_i{}' S_i^2$$

We can simplify the Hamiltonian to

$$H = H_J + H_V + H_\mu$$
$$= -J\sum_{\langle ij\rangle}{}' S_iS_j - V\sum_{\langle ij\rangle}{}' S_i^2S_j^2 + \Delta\sum_i{}' S_i^2$$

where $V = (V_{44} + V_{33} - 2V_{34})$

$$\Delta = \mu_3 - \mu_4 + \underset{\substack{\uparrow\\ \#\text{ nearest neighbors.}}}{z}(V_{33} - V_{34})$$

This is called the Blume-Emery-Griffiths model. ~~and~~ The model as Z_2 symmetry defined by $F S_i = -S_i$,
↳ flip.

The Z_2 symmetry can be broken (driven by the J term) and this can be measured by $M = \frac{1}{N} \sum'_i S_i$.

Note that $\left\langle \frac{1}{N} \sum'_i S_i^2 \right\rangle = 1-x$, is the concentration of ^4He. The question is what is the phase diagram of the system in the $T-\Delta-V$ space? We will work this out using meanfield theory.

$$H_{BEG}^{MFT} = -zJM\sum'_i S_i - z(1-x)V\sum'_i S_i^2 + \Delta \sum'_i S_i^2 + (\text{"non operator constant"}).$$

We can now obtain meanfield self consistency equation by using an standard approach. We ~~see~~ get

7)

$$M = \frac{2\sinh(\beta zJM)}{e^{\beta\tilde{V}} + 2\cosh(\beta zJM)}$$

$$(1-x) = \frac{\cancel{2\sinh}\ 2\cosh(\beta zJM)}{e^{\beta\tilde{V}} + 2\cosh(\beta zJM)}$$

$$\tilde{V} = \Delta + zV(1-x).$$

For a given J, V, Δ and T, we can now obtain M and (1-x) from the solution of the equations. We will now simplify the analysis by ~~no~~ taking V=0, (this assumes that interaction between particles is independent of the isotope - an excellent approximation in He).

Note that there can be multiple solutions to the mean field equations, and the solution with the least free energy is to be chosen. We'd like to ~~first~~ obtain the phase diagram in the T-Δ plane, and then from there infer the physics ~~at~~ in the T-x plane

Letus try to obtain physis when $\Delta = 0$. Here the selfconsistency eqns become

$$M = \frac{2\sinh(\beta zJM)}{1 + 2\cosh(\beta zJM)}$$

and

$$(1-x) = \frac{2\cosh(\beta zJM)}{1 + 2\cosh(\beta zJM)}$$

for $T > T_c$, we get $M=0$ and $x = 1/3$.
while for $T < T_c$ $M \neq 0$.

T_c can be obtained as

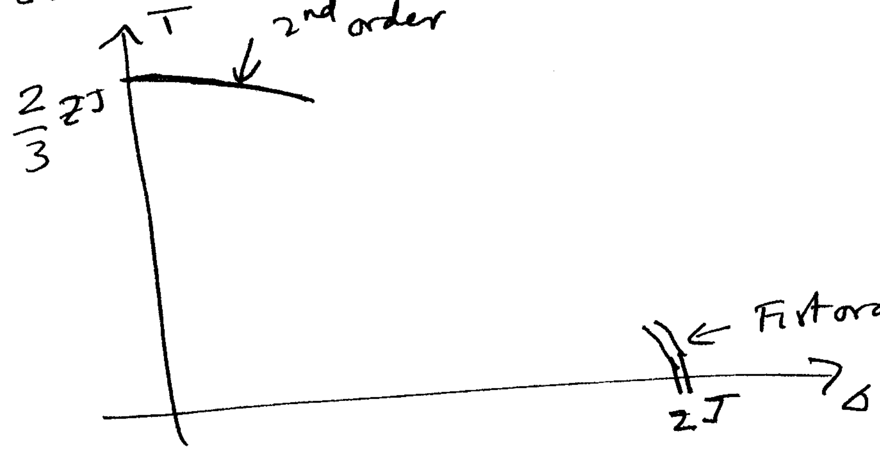
$$\beta_c zJ = \frac{3}{2} \Rightarrow \frac{T_c}{zJ} = \frac{2}{3} = 0.666$$

and this is a continuous transition,
For a small value of Δ, this will not change the nature of the transition. Note that as $T \to 0$ $x \to 0$, . Thus we see that in the T–Δ plane, for small Δ, we have a Second order line.

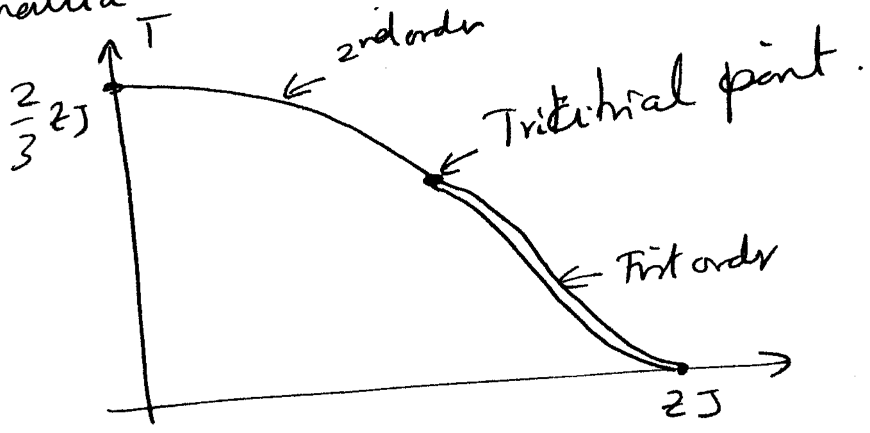
Now at $T=0$, we can study the system
The Energy levels are

S_i	E_i
1	$-zJM+\Delta$
0	0
-1	$zJM+\Delta$

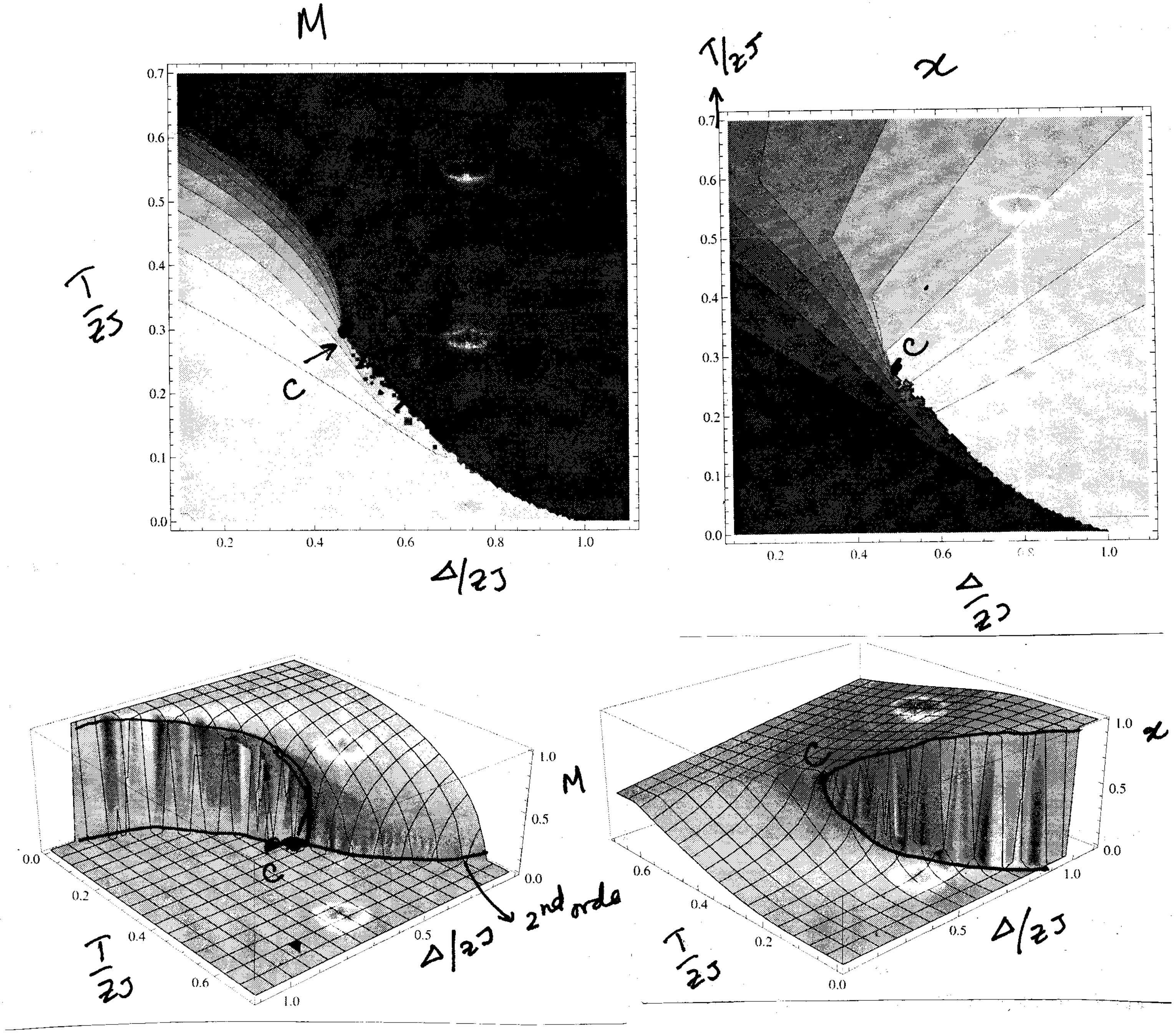
We see that until $\Delta < zJ$, the ground state has $M=1$, for $\Delta > \cancel{2}\, zJ$, the ground state is $S_i = 0$. ~~The concl~~ This is a first order transition. ~~Thus~~ This story does not change for a small $T>0$. Our conclusion is that



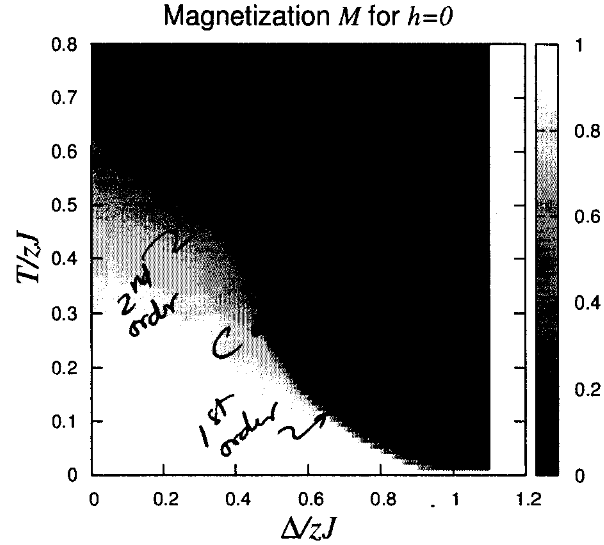
To find the full phase diagram we use Mathematica and obtain

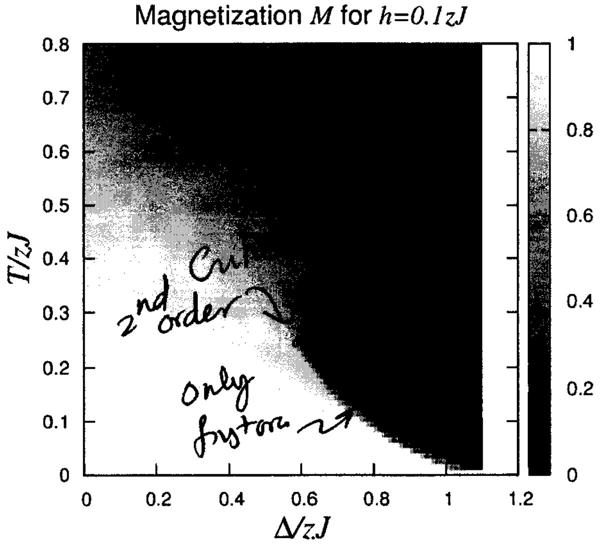


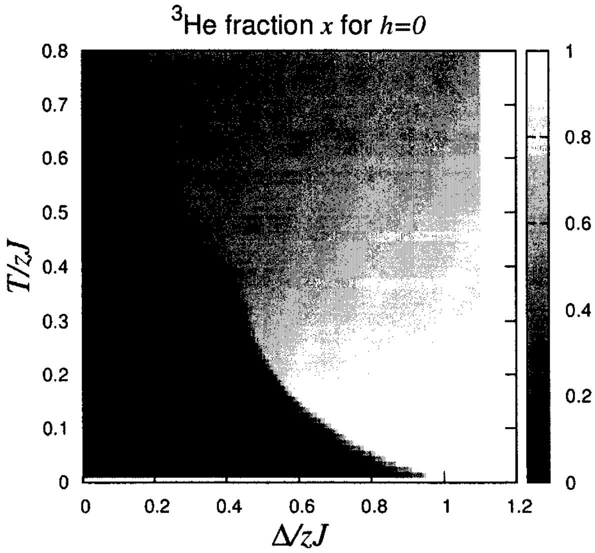
This example illustrates a <u>tricritical point</u> in the phase diagram. At such a point a first order line ends along with a second order line.
Detailed ~~N~~ Numerical results are shown in the next page.

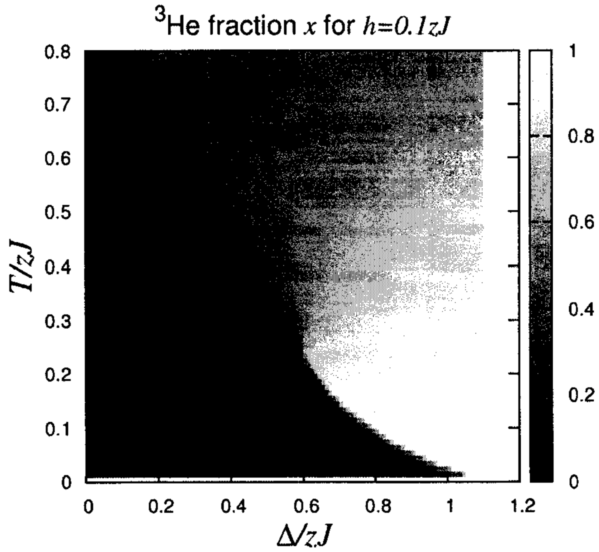


Numerical results for the BEG model with V=0.

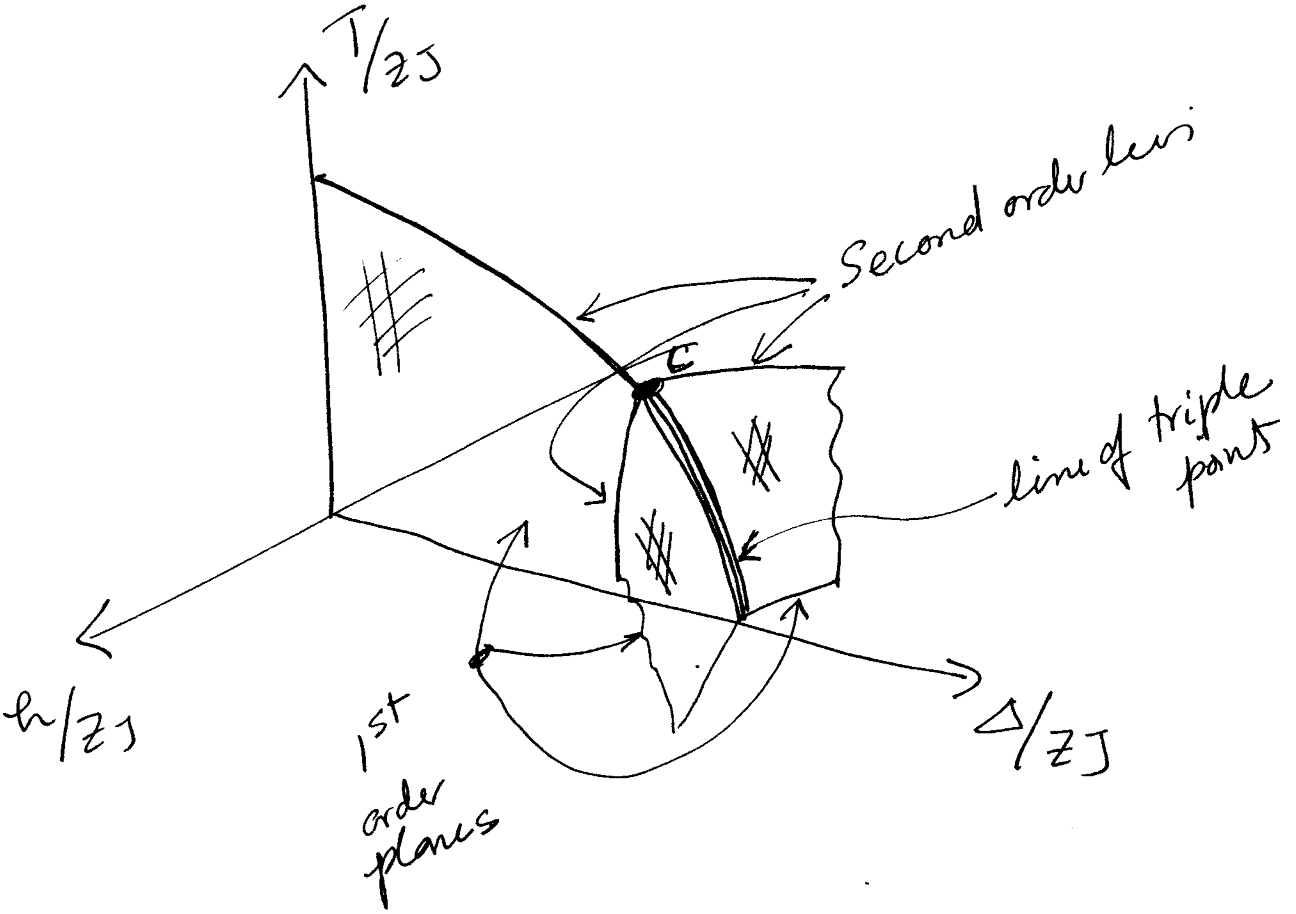








The figure in the top left panel is same as that plotted on the previous page with $h=0$. On the top right we show magnetization, ~~[illegible] fraction~~ We see that the second order line is now gone, and there is only a first order line that terminates at a second order point (Just as in the Ising model).



The phase digram is made of

① Three first order phases marked by ###

② Three second order lines which meet at C

③ A line of triple points which ends at C.

One can see that there are very rich possibilities for phase digrams!

Since three second order lines meet at C it is called a tricritical point.
Bicritical and tetra critical point are defined analogously

Landau (and Ginsburg) Theory of Phase Transition.

Let us briefly recall what we have done so far. We said that systems are in diffrent phases under diffrent conditions From a microscopic point of view, we can obtain the phase diagram of the system by calculating the density matrix

$$\rho(x) = \frac{e^{-\beta \mathcal{H}(x)}}{Z}$$

where we have always understood that we are applying a symmetry breaking field, taking the thermodynamic limit, and then taking the field to be zero. We also want that a broken symmetry phase is described by an order parameter such as $\mathcal{M}$; $M = \langle \mathcal{M} \rangle$. Note that $\mathcal{M}$ is ~~not zero~~ is not an invariant operator I..e, If U is a symmetry $U^{+} \mathcal{M} U \neq \mathcal{M}$.
The key point ~~was~~ Landau realized was that We can study phases and

phase transitions by directly looking at the order parameter. Stated in other words, the ordered phase has a non vanishing order parameter. - Can we develop a theory to find conditions under which the equilibrium value of the order parameter M becomes non zero? Landau realized that the order parameter has ~~to~~ be elevated to a thermodynamic variable and one needs to write a free energy directly ~~in~~ in terms of the order parameter M ~~there~~. Let us call this $f_L(T, M)$ T is the temperature (but could ~~include~~ any other parameter controlled in experiment). To see whether we are in an ordered (broken symmetry phase), ~~How~~ Solve for the equilibrium value of M

$$\frac{\partial f_L(T, M)}{\partial M} = 0 \rightarrow \text{gives } M \text{ at equilibrium}$$

If $M_{equ} \neq 0$, we are in the ordered phase, while $M_{equ} = 0$ we are in the disordered phase.

~~Note that~~ How do we obtain $f_L(T,M)$ (f_L is the free energy density).

There are two approaches:

① Microscopic approach

For our system with Hamiltonian $H(x)$, Define a new Hamiltonian with a symmetry breaking field

$$H_h(x) = \mathcal{H}(x) - h\, M(x).$$

$$\rho_h(x) = \frac{e^{-\beta H_h(x)}}{Z}.$$

Now $F(T,h) = -T \ln \mathrm{tr}\, \rho_h$

this is the free energy.

We can now obtain the Landau free energy by a Legendre transform. (This is analogous to changing ensembles, i.e, for example going from the Helmholtz to Gibbs free energies).

$$M = -\frac{\partial F(T,h)}{\partial h}.$$

In other words

$$dF = -SdT - Mdh$$

Now define

$$G(T, M) = F + Mh$$

$$G = dF + d(Mh)$$

$$dG = -SdT - Mdh + Mdh + dM\,h.$$

$$dG = -SdT + hdM.$$

$$\left.\frac{\partial G}{\partial M}\right|_T = h. \; !$$

~~Suppose~~ We now see that $f_L(T,M)$ is actually equal to $G(T,M)$.

$$f_L(T,M) = \underbrace{F(T,h) + Mh}_{\hookrightarrow \text{Legandre transform.}}$$

~~The~~ To find spontaneous symmetry breaking we need to set

$$\frac{\partial f_L(T,M)}{\partial M} = 0.$$

and solve this equation for M. If the smallest f_L is obtained for $M \neq 0$, then we say that the ~~spontaneous~~ symmetry is spontaneously broken!

$f_L(T,M)$ is also called the Coleman-Weinberg potential in high energy physics.

But as you can see, this is all "theory", since we cannot obtain $F(T,h)$ as this is a very hard problem. We can certainly do this approximately by making ansatz such as mean field theory. Infact the meanfied free energy in terms of $M = \langle S_i \rangle$ (for the Ising model) <u>is</u> an example of $f_L(T,M)$. The discussion so far suggests the key ideas, although, in general it may not be possible to follow these ideas from a microscopic approach. What to do now - well, ask Landau!

2) <u>Phenomenological Approach.</u>

Landau realized that the calculation of $f_L(T,M)$ from microscopics is a hopeless task. So he decided to do what he does best - think! Landau noted a key property of $f_L(T,M)$. Suppose U is a symmetry of the system that is spontaneously broken in the ordered state. then $UM \neq M$. Now since U is a symmetry of the system

$$f_L(T, UM) = f_L(T, M). \quad -(1).$$

In other words $f_L(T,M)$ must respect the symmetry of the system.

Now Landau ~~made~~ took the next crucial step. He Crucial step ~~th~~ insisted the for a given T (~~or~~ any any other parameters), the function $f_L(T,M)$ is an *analytic function* of M. Note that M may be any kind of quantity such as scalar, vector, ~~or~~ a tensor. ~~Let~~. To continue the discussion let us be more specific. Let us assume that our system has $O(n)$ symmetry & ~~O(1)~~ this is the $O(1) \equiv Z_2$ (by definition), this is like the Ising symmetry, $O(2) \to$ this is like the X-Y model, $O(3)$, the Heisenberg model etc.

For $O(n)$ symmetry M would be an

n-component vector which we will call as $\vec{M}$. The conjugate field will be the vector $\vec{h}$.

Now ~~[illegible]~~ the two condition, namely the invariance of f_L to $\mathcal{U}$ and the analyticity ~~such~~ suggests that

$$f_L(T,\vec{M}) = \sum_n \overset{u_n(T)}{\cancel{a_n(T)}}\, P_n(\vec{M})$$

where $a_n(T)$ are analytic functions of T and $P_n(\vec{M})$ are $\mathcal{U}$ invariant analytic functions of $\vec{M}$.

For $O(\vec{n})$ symmetry a $\mathcal{U}$ ~~[illegible]~~ $\in O(n)$ invariant analytic function is $|\vec{M}|^2$ or $|\vec{M}|^4$ or $|\vec{M}|^{36}$ ..etc.

Then

$$f_L(T,\vec{M}) = \sum_{n=1}^{\infty} \overset{u_{2n}}{\cancel{a_{2n}}}(T)\,|\vec{M}|^{2n}$$

$$= \overset{u_0}{\cancel{a_0}}(T) + \overset{u_2}{\cancel{a_2}}(T)\,|\vec{M}|^2 + \overset{u}{\cancel{a}}_4(T)\,|\vec{M}|^4$$

Suppose I have an external field, then

$$f_L(T,\vec{h};\vec{M}) = f_L(T,\vec{M}) - \vec{h}\cdot\vec{M}.$$

The equilibrium value of $\vec{M}$ is obtained by

$$\frac{\partial f_L(T,\vec{h};\vec{M})}{\partial \vec{M}} = 0.$$

~~If we put $\vec{h}=0$~~ We get

$$\frac{\partial f_L(T;\vec{M})}{\partial \vec{M}} = \vec{h}.$$

~~If~~ We put $\vec{h}=0$, and solve this for $\vec{M}$. Suppose we find that $\vec{M} \neq \vec{0}$, then we will have a ~~p~~ spontaneous broken symmetry..

Let us do this for the Ising model. $O(1) \equiv Z_2$; and this $\mathcal{U} \in Z_2$.

$$f_L(T,M) = \underset{\substack{\uparrow \\ \text{analytic} \\ \text{part of free eng}}}{f_0(T)} + \frac{u_2(T)\,M^2}{2} + \frac{u_4(T)\,M^4}{4}.$$

Now what are $f_0(T)$, $u_2(T)$ and $u_4(T)$? (Note that factors 2 and 4 are introduced for convenience)

First thing, ignore $f_0(T)$ as this is smooth (analytic). Now choose

$$u_2(T) = a(T-T_c)$$ where $a>0$

and T_c ~~tis~~ is an unknown transition temperature.

$$u_4(T) = b > 0.$$

$$f_L(T,M) = \frac{a(T-T_c)}{2} M^2 + \frac{b}{4} M^4.$$

Note that the choice of $u_2(T)$ and $u_4(T)$ are made to "explain an observation" - ~~these~~ these choices are phenomenological

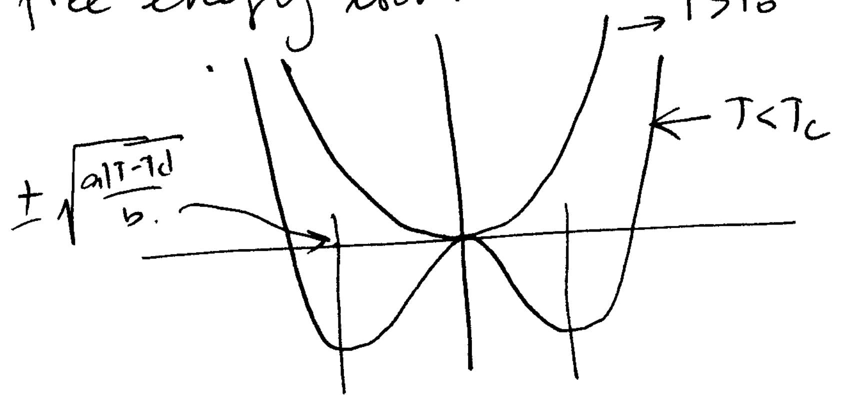
Now let us see what is obtained

$$\frac{\partial f_L}{\partial M} = 0 \Rightarrow a(T-T_c)M + bM^3 = 0.$$

This means that the equilibrium value of M is

$$M = \begin{cases} 0 & T > T_c \\ \pm\sqrt{\dfrac{a|T-T_c|}{b}} & T \le T_c. \end{cases}$$

The free energy looks like

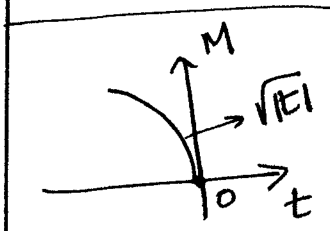


Landau theory therefore predicts that

$$M(T) = \pm\sqrt{\frac{a}{b}}\sqrt{|T-T_c|}$$

There is a more standard way of writing this. Introduce $t = \frac{T-T_c}{T_c}$

Then $M(T) = \pm\sqrt{\frac{aT_c}{b}}\sqrt{|t|}$.



$$= \pm\sqrt{\frac{aT_c}{b}}\,|t|^{\beta} \qquad \boxed{\beta = 1/2}$$

β is called a critical exponent and its value predicted by Landau theory is 1/2.

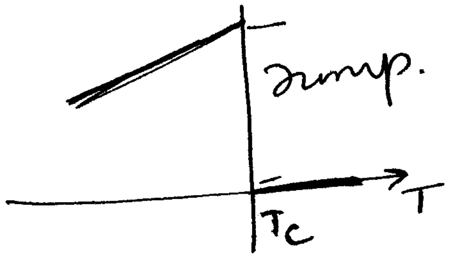
Now what about the specific heat.

$$C_V = -T\frac{\partial^2 f_L}{\partial T^2} \simeq -\frac{T}{T_c^2}\frac{\partial^2 f_L}{\partial t^2}$$

$$f_L = \begin{cases} 0 & T > T_c \\ \cancel{\frac{1}{2}\frac{a^2T_c^2}{b}(-t)^2 + \frac{b}{4}\frac{a^4T_c^2}{b^2}(-t)^2} & \end{cases}$$

$$= \begin{cases} \frac{aT_c t}{2}\cdot\frac{a}{b}(-t)T_c + \frac{b}{4}\frac{a^2T_c^2}{b^2}t^2 \\ = \frac{-aT_c^2}{4b}t^2 & T<T_c \end{cases}$$

$$C_V = \begin{cases} 0 & T > T_c \\ \dfrac{aT}{2b} & T < T_c \end{cases}$$



Landau theory predicts a <u>jump</u> in C_V

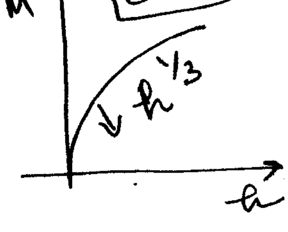
$C_V \sim |t|^{-\alpha}$ $\quad \alpha = 0$ by Landau theory.

We can now calculate the magnetization in a magnetic field h

We get

$$a(T-T_c)M + bM^3 = h$$

$\boxed{\delta = 3}$



at $T = T_c$ we see that

$\boxed{\delta = 3}$ $\quad M = \frac{1}{b} h^{1/3} \sim h^{1/\delta}$

at T_c the response of system is ~~[illegible]~~ <u>non linear</u>. For $T < T_c$

$$M = \cancel{M_0 +} \; \text{sign}(h)\,(M_0) + \Delta M\}$$

$$\text{sign}(h)\,(M_0) + \Delta M\}$$

where Δm is the additional magnetization due to h.

We see that

$$a(T-T_c)\left[\text{sign}(h)(M_0)+(\Delta m)\right] + \cancel{\text{Sign}\; b(M_0^3+3M_0^2\Delta m)}\ \cancel{= h}$$

$$+\ b\left[\cancel{\text{Sign}}\,\text{Si}(h)M_0+\Delta m\right]^3$$

$\Rightarrow$

$$\Rightarrow \quad aT_c\, t\, \Delta m + 3bM_0^2\Delta m = h$$

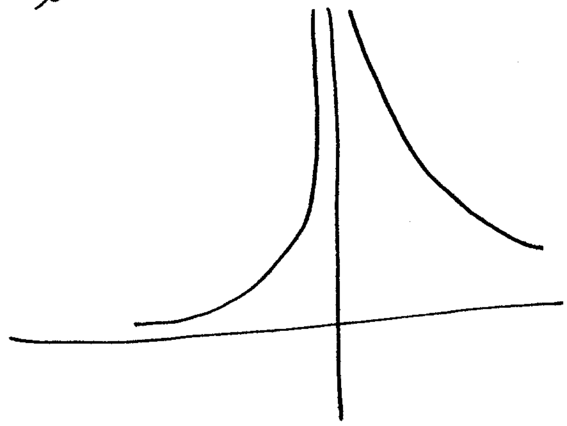
For $t>0 \qquad M_0^2 = 0$

$$\Rightarrow \chi = \frac{\Delta m}{h} = \frac{1}{aT_c}\,\frac{1}{t}$$

For $t<0 \quad ,\ bM_0^2 = -aT_c t$

$$\Rightarrow \quad \chi = -\frac{1}{2aT_c}\,\frac{1}{t}.$$

We see that χ diverges near t_c



$$\chi = C_\pm \frac{1}{|t|}$$

$$= C_\pm\, |t|^{-\gamma}$$

$\gamma = 1$

Note $\dfrac{C_+}{C_-} = 2$

Thus we get that the critical exponent $\gamma = 1$

What have we learnt from Landu theory? Based on two ideas (a) Invariance of the free energy to the symmetries of the system and (b) analyticity of the free energy, on the order parameter, we have predicted values of for <u>critical exponent</u> and the amplitude ratios

$$\alpha = 0$$

$$\beta = 1/2$$

$$\gamma = 1 \qquad \frac{C_+}{C_-} = 1/2$$

$$\delta = 3$$

~~Note~~ Note that these results are quite interesting. For,

① Only the Ising symmetry and the presence of a T_c enters the formulation of $f_L(T, M)$

② The spatial dimension <u>does not</u> enter the discussion any where.

~~This means~~ Ladau theory predicts that the Ising critical exponents are independent of spatial dimensions. How well does this hold up with known results?

Results for the Ising model

quant	$d=1$	$d=2$	$d=3$	$d=4$	Landau
T_c	0	2.27 J	4.51 J	6.68J	nothing to say
α	1	0	0.110	0	0
β	0	1/8	0.326	1/2	1/2
γ	1	7/4	1.237	1	1
δ	∞	15	4.789	3	3.

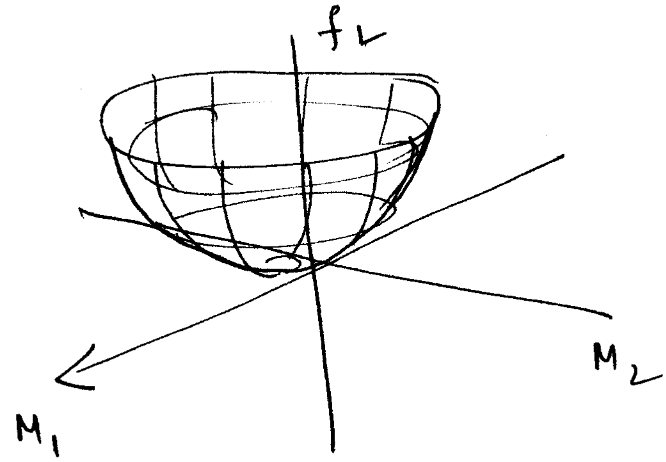
~~In fact there really exist known to be true for all $d>$~~ We see something absolutely remarkable! $d \leq 3$ the critical exponents are strongly d dependent. For $d = 4$ (and all $d>4$) the critical exponents are same as that predicted by Landau theory! Landau must be getting something right!

Now we ask - ~~can~~ what about systems with other symmetries; ~~we~~ like for example $O(n)$ symmetry with $n>1$. For $n=2$, we know that this describes the symmetry of the XY model, and we have experimental results from superconductors on ^4He to compare with.

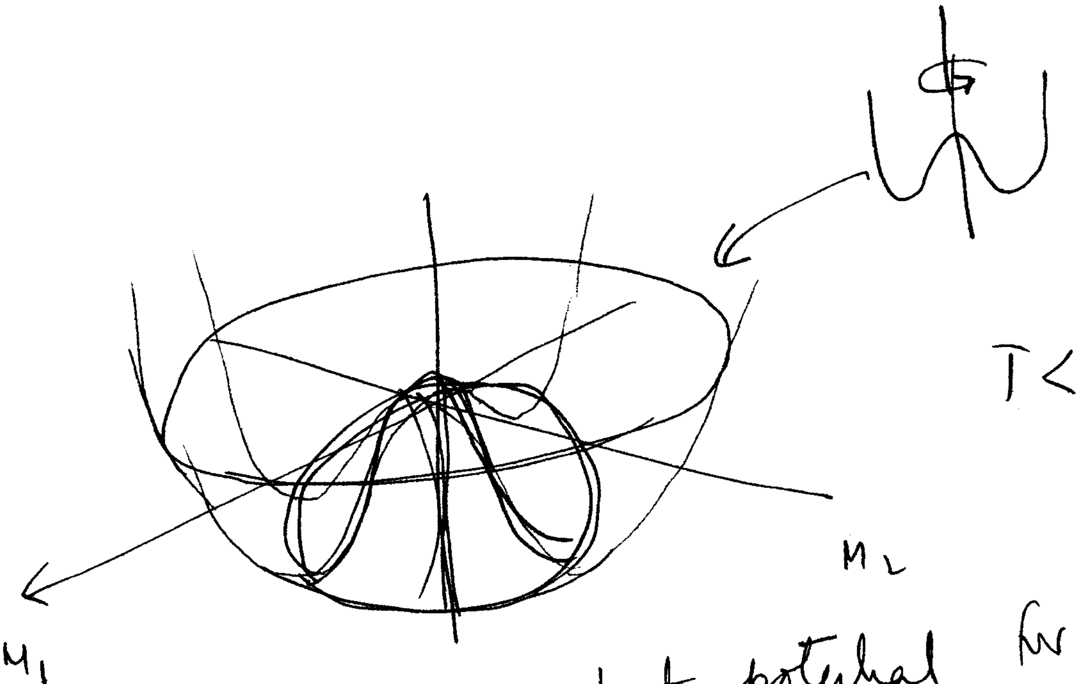
Here $\vec{M}$ is a vector and by argument quite similar to what we followed

$$f_L(T,\vec{M}) = \frac{a(T-T_c)}{\cancel{2}}|\vec{M}|^2 + \frac{b}{4}|\vec{M}|^4$$

We can plot the free energy for O(2) above and below T_c



$T > T_c$



$T < T_c$

We have a Mexican hat potential for $T < T_c$.

~~For T < Tc~~

The equilibrium value of $\vec{M}$ ~~for Tc~~

is
$$\vec{M} = \begin{cases} \vec{0} & T > T_c \\ |\vec{M}|^2 = \dfrac{T_c a |t|}{b} & T < T_c . \end{cases}$$

We see that $\beta = 1/2$.

Note that $|\vec{M}|$ is what is determined, the system can chose any direction in $\vec{M}$ space!

By the same argument that were used ~~to arrive~~ for the Z_2 case, we see that $\alpha = 0$

The suceptibility is a bit more complicated

We have
$$a T_c t \vec{M} + b \vec{M} |\vec{M}|^2 = \vec{h} .$$

For $t > 0$ $(T > T_c)$
$$\chi = \frac{1}{a T_c} \frac{1}{t} .$$

For $t < 0$ $T < T_c$.

~~We~~ write $\vec{M} = \vec{M}_0 + \Delta\vec{M}$ where $\vec{M}_0$ is the spontaneous symmetry broken value (also called "vaccum expectation value".)

$$\Rightarrow \quad aT_c t\,(\vec M_0 + \Delta\vec M) + b(\vec M_0 + \Delta\vec M)(M_0^2 + 2\vec M_0\cdot\Delta\vec M) = \vec h$$

$$\Rightarrow \quad aT_c t\,\Delta\vec M + bM_0^2\,\Delta\vec M + 2b(\vec M_0 \otimes \vec M_0)\,\Delta\vec M = \vec h \qquad (▨)$$

Let $\hat n \equiv$ direction of $\vec M_0$

$$aT_c t\,\Delta\vec M\cdot\hat n + bM_0^2\,\Delta\vec M\cdot\hat n + 2b\vec M_0^{2}\,\Delta\vec M\cdot\hat n = \vec h\cdot\hat n \qquad (★)$$

~~$\chi_\parallel$~~ $$\underbrace{(aT_c t + 3bM_0^2)}_{\chi_\parallel^{-1}}\,\underbrace{\Delta\vec M\cdot\hat n}_{\Delta M_\parallel} = \underbrace{\vec h\cdot\hat n}_{h_\parallel}$$

$$\Rightarrow \quad \chi_\parallel^{-1} = -\frac{1}{2aT_c}\,\frac{1}{t} = \frac{1}{2aT_c}\,\frac{1}{|t|}$$

~~$\chi_\perp$~~ To calcuate $\chi_\perp$ subtract ★ from ▨ to obtain

$$aT_c t\,(\underbrace{\Delta\vec M - (\Delta\vec M\cdot\hat n)\hat n}_{\Delta\vec M_\perp}) + bM_0^2\underbrace{(\Delta\vec M - \Delta\vec M\cdot\hat n\,\hat n)}_{\Delta M_\perp} = \underbrace{\vec h - (\vec h\cdot\hat n)\cdot\hat n}_{\vec h_\perp}$$

We see that

$$(aT_c t + bM_0^2)\,\Delta\vec M_\perp = \vec h_\perp$$

$$\Rightarrow \quad \chi_\perp = \infty! \qquad T < T_c.$$

This is not surprising since if we apply ~~to ⊥ field to a broken symmetry~~ a field perpendicular to the broken symmetry direction all the spins will point in the direction of the applied field with no additional cost in energy..

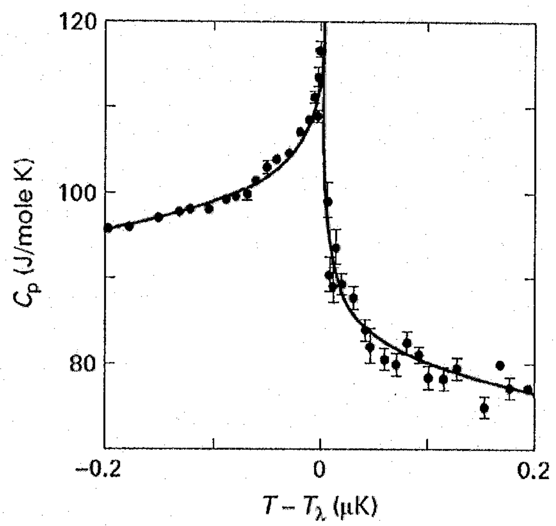
These considerations suggest that

$$\chi = \begin{cases} \chi = \frac{1}{aT_c}\frac{1}{t} & T>T_c \\ \chi_{\parallel} = \frac{1}{2aT_c}\frac{1}{|t|} & T<T_c . \end{cases}$$

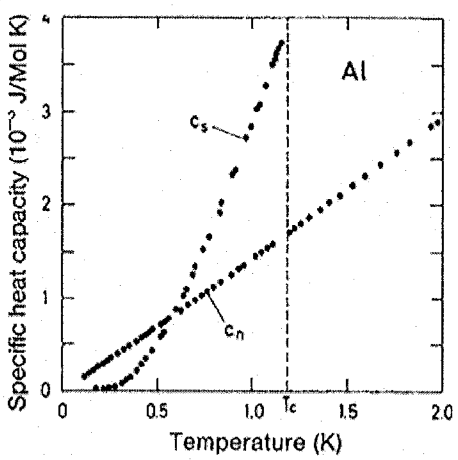
and ~~$\vec{M}_0$ at T_c~~ $\vec{M} = \frac{1}{b}\hat{h}\,|h|^{1/3}$

Hence $\gamma = 1$ and $\delta = 3$. Our conclusions are stark.

$\alpha = 0$, $\beta = 1/2$, $\gamma = 1$, $\delta = 3$ in Landau theory independent of n! and d!!. All second order critical points are same within Landau theory!! This is a stricking prediction and is immediately amenable to checks.



Experiments on the *SPACE SHUTTLE* for specific heat of the superfluid transition in helium, shows that $C_v = |T - T_\lambda|^{-0.0127}$



Experimental specific heat Al at the superconducting transition. This shows a jump (c_s – superconducting, c_n-normal state) in specific heat.

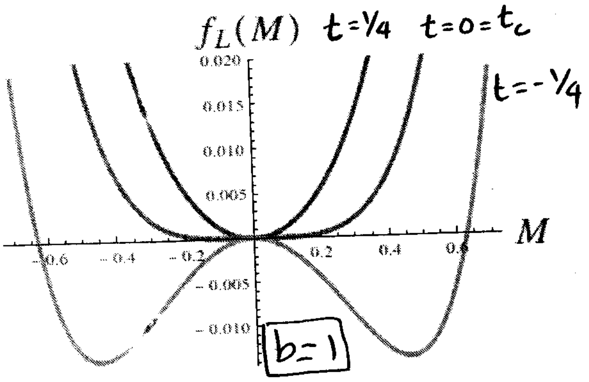
Now ^{4}He shows a divergence of specific heat, while the superconducting transition shows a jump! This is very strange since both these transition have the same broken symmetry. What is clear is that experiments (and calculation) are at variance with the predictions of the Landau theory! Yet Landau theory seems "so correct"! What shall we do to reconcile these thing?

Before we address this question, I would like to call your attention to ~~the~~ an even more pressing question! Our usual Ising model has Z_2 symmetry and the BEG model (and the Blume-Capel model) also has ~~Ising~~ Z_2 symmetry. The BEG/BC models have tricritical points! How can we understand this from Landau theory? Note that Landau theory is <u>phenomenological</u>, in other words, we can use the "tricritical phenomenology" to obtain a phase diagram similar to the BC model. A key phenomenological observation is that the tricritical point needs $\{T, \Delta\}$ and h to be chosen [illegible]. Thus our Landau theory for the Ising model has only a single parameter t ~~and th~~ (two if we also include h), and therefore <u>cannot</u> do the job. What should we do? We have to, of course, ~~conside~~ consider a more sophisticated free energy function.

Since our system has Ising (Z_2) symmetry, our order parameter is a real number M

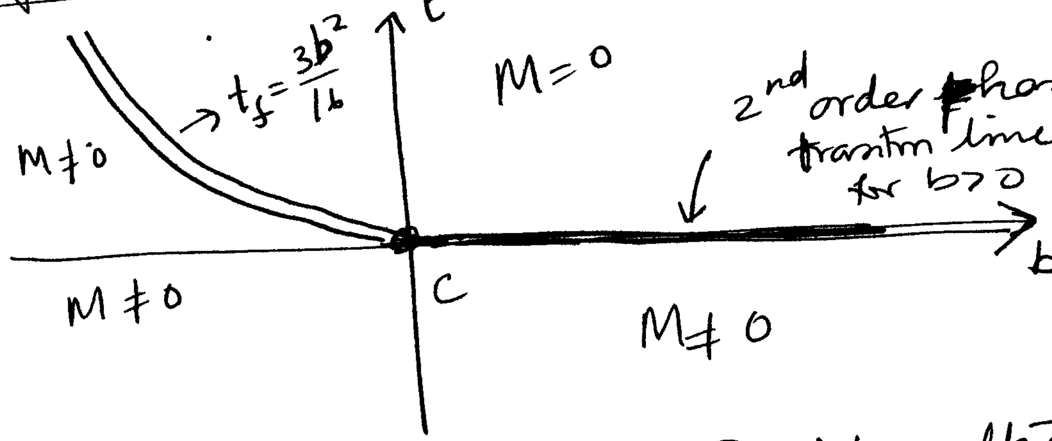
Consider
$$f_L(t, b; M) = \frac{t}{2} M^2 + \frac{b}{4} M^4 + \frac{c}{6} M^6$$

with $c > 0$. (a fixed quantity). Note that there are two "external" parameters t and b. We would therefore like to obtain a ~~pose~~ phase diagram in the t-b plane. ~~Consider b>0~~. Consider $b > 0$, Then the quantic and hextic term are positive definite. The free energy can have multiple stationary points only when $t \leq 0$, ~~bed~~ beginning with $t = 0$. This can be seen by a plot of the free energy.

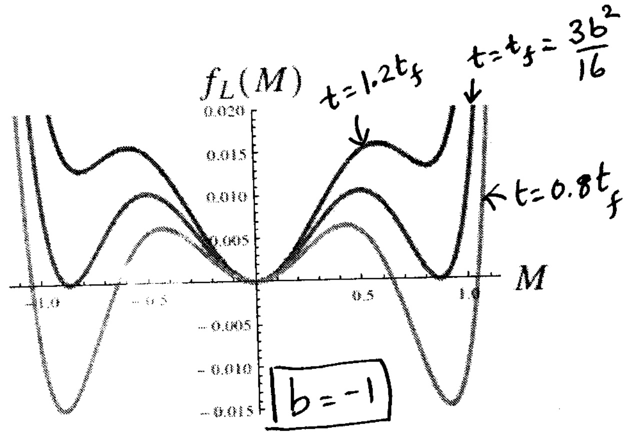


We see that for $b > 0$, $t = 0$ is a

Second order point - One can also show this explicitly by ~~the~~ minim of free energy. ~~what~~ ~~has~~ Thus $(b>0, \; t=0)$ as a line of second order point.



what happes when $b<0$? Now the free energy has multiple minima when $t>0$. See the figure on the right. In fact for a critical value ~~of t_f value~~ $t=t_f$, the minima at $M\neq 0$ has a value equal to that a $M=0$. This is a first order transition, for $t<t_f$, $M\neq 0$ ~~has~~ at the minimum. We can show that $t_f(b) = \frac{3b^2}{16}$. The point $C=(b=0, t=0)$ is a tricritical point. We clearly see that Landau theory ~~has~~ can indeed capture complex physics such as that ~~the~~ ~~pha~~ of the tricritial point.



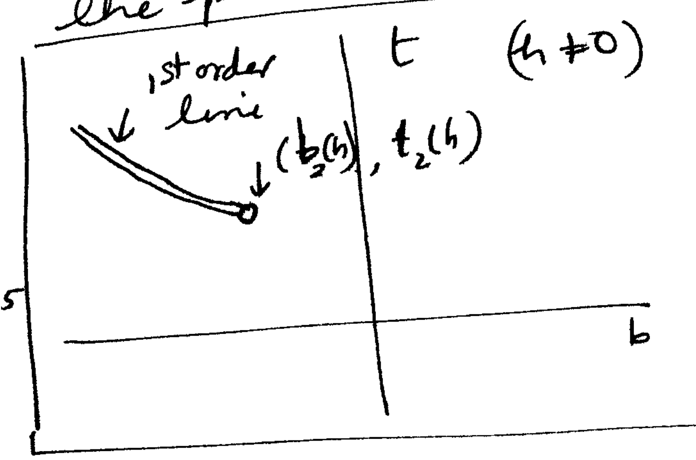
We can even get a detalied understanding of the tricritically. To see this include a magnetic field h, and write

$$f_L(t,b,h;M) = f_L(t,b;M) - hM.$$

The extrema of free energy satisfies

$$tM + bM^3 + cM^5 - h = 0.$$

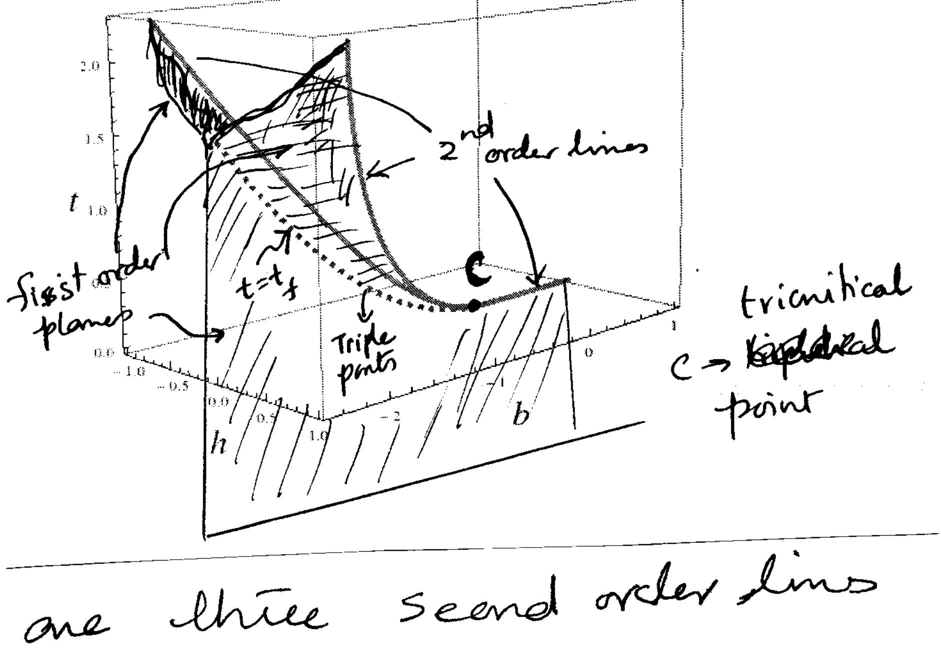
An analysis of this condition shows that for a given $h \neq 0$, the phase digram looks like shown on the right. A first order line terminates at a critical point (2nd order)



$$b_2(h) = -\frac{5c^{3/5}(h^2)^{1/5}}{2^{1/5}\,3^{1/5}}$$

$$t_2(h) = \frac{5\,3^{4/5}\,c^{1/5}(h^2)^{2/5}}{4\,2^{2/5}}.$$

The full phase digram is shown below.



Three are three second order lines

$(0, b_3 > 0, 0)$, $(b_2(h), t_2(h), h)$ and $(b_2(h), t_2(h), -h)$ which all meet at $C = (b=0, t=0, h=0)$ which is a tricritical point. The line ~~t=~~ $(b_3 < 0, \frac{3b^2}{12}, 0)$ is a line of triple points. Three planes of first order transition end on this triple point line. One sees that the topology of the phase digram is ~~exact~~ identical to that of the BC model (obtained by mean field Landau theory). Quite clearly Landau theory is quite powerful. ~~and~~

We can even study the properties of the tricritical point C.

At the tricritical point if we vary t we get

$$t M + \overset{b=0}{\cancel{bM^3}}{}^{\to 0} + c M^5 - h = 0 .$$

For $h = 0$, we get $M_0^4 = -\frac{t}{c}$

or $M_0 = \left[-\frac{t}{c}\right]^{1/4}$ giving us $\beta = 1/4$.

We get that

$$f_L^{\min}(t, b=0, h=0) = \begin{cases} 0 & t > 0 \\ \underbrace{\frac{t}{2}\left(-\frac{t}{c}\right)^{1/2} + \frac{c}{6}\left(-\frac{t}{c}\right)^{3/2}}_{-\frac{1}{3\sqrt{c}}(-t)^{3/2}} & t < 0 \end{cases}$$

We get $C_V = -T \dfrac{\partial^2 f_L}{\partial t^2}$

$$= -\frac{T}{T_c^2}\left[\begin{array}{ll} 0 & t>0 \\ \dfrac{1}{4\sqrt{c}}\dfrac{1}{(-t)^{1/2}} & t<0 \end{array}\right.$$

$$\Rightarrow \quad \alpha = 1/2$$

Now at $t=0,\ b=0$

$$M \propto \left(\frac{h}{c}\right)^{1/5} \Rightarrow \quad \delta = 5$$

when $t \neq 0$ $\quad M = M_0 + \Delta M$

$$\Delta M = \frac{h}{t} \qquad t>0$$

for $t<0$ $\quad (t + 5cM_0^4)\,\Delta M = h$

$$\Rightarrow \quad -4t \cdot \Delta M = h$$

or $\quad \Delta m = \dfrac{h}{-4t}$

$$\Rightarrow \quad \chi = \left\{\begin{array}{ll} 1/t & t>0 \\ -\dfrac{1}{4t} & t<0 \end{array}\right.$$

$\Rightarrow \quad \gamma = +1$ and $\dfrac{C_+}{C_-} = 4$. (the amplitude ratio).

We learn something quite interesting. The critical exponents of the tricritical point are diffrent from that of the usual critical point! Furthermore there will be additional critical exponents such as that obtained by setting $t=0$, and varying b

For example $M_0(0,b) = \begin{cases} 0 & b>0 \\ \sqrt{-\frac{b}{c}} & b<0. \end{cases}$

and so on. What is important is that Landau theory can predict these, and this can be checked with experiments.

We are now reassured that Landau theory can provide a framework to "understand" phase diagrams of systems based on ideas of symmetry and analyticity of the free energy on the order parameter.

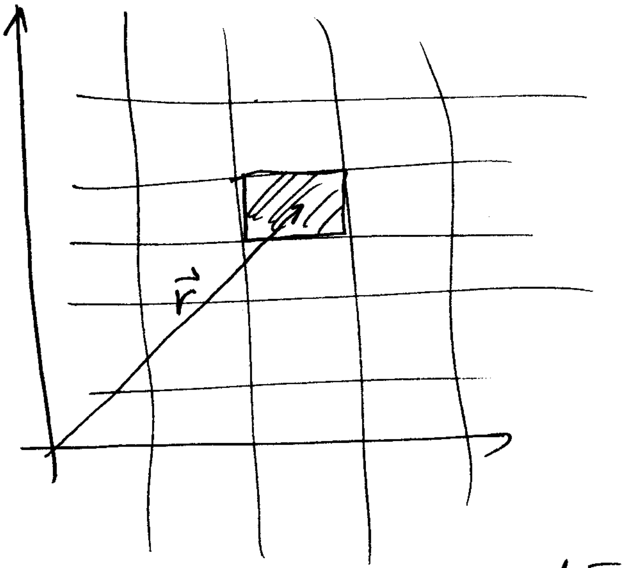
We therefore need to address the central question:

① ~~why does~~ Landau theory does not depend on the spatial dimension of the system? ~~Whose~~ Moreover in "high" dimension Landau theory does remarkably well and infact $d \geq 5$, it is *exact*!

② Landau theory is essentially independent of the symmetry of the order parameter. This is clearly wrong as is known from experiment.

Question is: What is missing in Landau theory?

Let us look at Landau's ideas from a diffrent prespective



Imagine that ~~we~~ ~~our~~ system is very large and can be chopped up into sub systems ~~which~~ which are connected with each other as shown above. For example, the subsystem ~~r~~ shown in the figure is labelled by the vector $\vec{r}$. In thermal equilibrium at temperature T, Landau theory assumes that the order parameter is *uniform*, i.e. $M(\vec{r}) = M$, and in each subsystem this defines a free energy density as fn. of M using symmetry and analyticity. Quite intersting this is *exactly* what we did in meanfield theory; ~~tau~~ and ~~if~~ in fact the results of MFT and Landau theory are identical! ~~th~~ In this sense, Landau theory is a "smart" way of doing meanfield theory without any microscopics - or better still, Landau theory gives

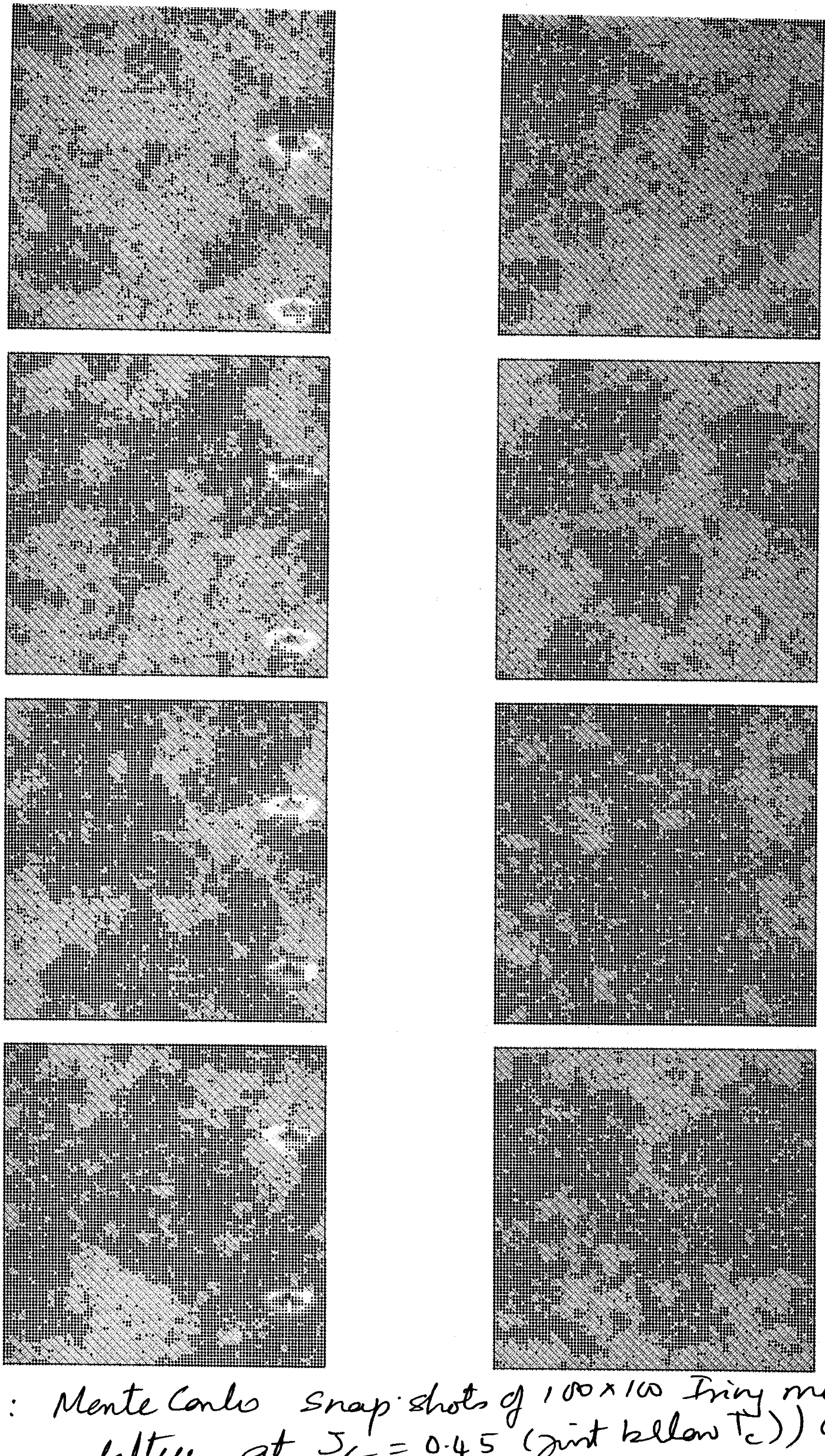


Fig: Monte Carlo snap-shots of 100×100 Ising model (square lattice at $J/T = 0.45$ (just below T_c)) One sees large fluctuations

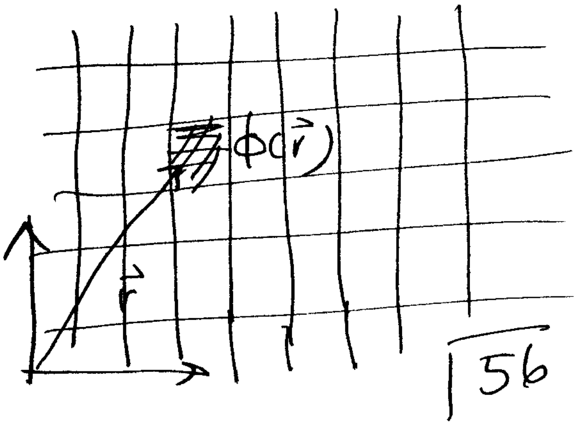
a way to obtain the form of meanfield free energy.
We know what is missing in meanfield theory
It is FLUCTUATIONS! This can be realised

looking at snapshots of spin configuration in a Montee Carlo calculation. Just below T_c, Landau theory would predict a magnetization $M \sim \sqrt{t}$. (for the 2d-square lattice Ising model). In the MC we see that there are lots of fluctuations, and infact these fluctuations occur at various length scales. All of this stuff is missed out by Landau theory. This leads to the natural question – what is the "cheapest" way to include fluctuations? Enter: Ginzburg!

Landau and Ginsburg decide to include fluctuations in their theory by ‡ making the following steps.

① They first made the order parameter a function of space. The fluctuating order parameter field is called $\vec{\phi}(\vec{r})$. (We have assumed that there is $O(n)$ internal symmetry). Now visualize the system as a large collection of subsystems which are interconnected

The free energy of the order parameter variable in the sub-system is



The free energy of the total system is

$$\int d^d\vec{r} \;\; f_L(T; \vec{\Phi}(\vec{r}))$$

where f_L is the Landau free energy density that was put-down analytically. But there is an issue with this. formulation. How does the order parameter in one cell "know" about the order parameter in another cell. Surely, they would like to be "same" and if they are diffrent, there must be an energy penalty. LG introduced the energy penalty, again by symmetry and analyticity! The diffence between the order parameters in the diffrent cells ~~need~~ is $\vec{\nabla}\phi(\vec{r})$. We need a scalar that respects all the symmetries of the system. For an isotropic system we have

$$\cancel{\frac{K}{2}} \int d^d\vec{r} \;\; \frac{K}{2} \left| \vec{\nabla}\vec{\Phi}(\vec{r}) \right|^2$$

~~note~~ which is both analytic and respect the symmetries (O(n)). The total gives us the famous Landau - Ginzburg

Hamiltonian functional

$$\beta H_{LG}[\vec{\phi}(\vec{r})] = \int d^d r \ \mathcal{H}_{LG}(T_2 ; \vec{\phi}(\vec{r}))$$

$$\mathcal{H}_{LG}(T; \vec{\phi}(\vec{r})) = \frac{K}{2} |\vec{\nabla}\vec{\phi}(\vec{r})|^2 + f_L(T; \vec{\phi}(\vec{r}))$$

Note that the partition function of the system is obtained by tracing ~~over~~ (summing) over all possible configurations of the order parameter field.

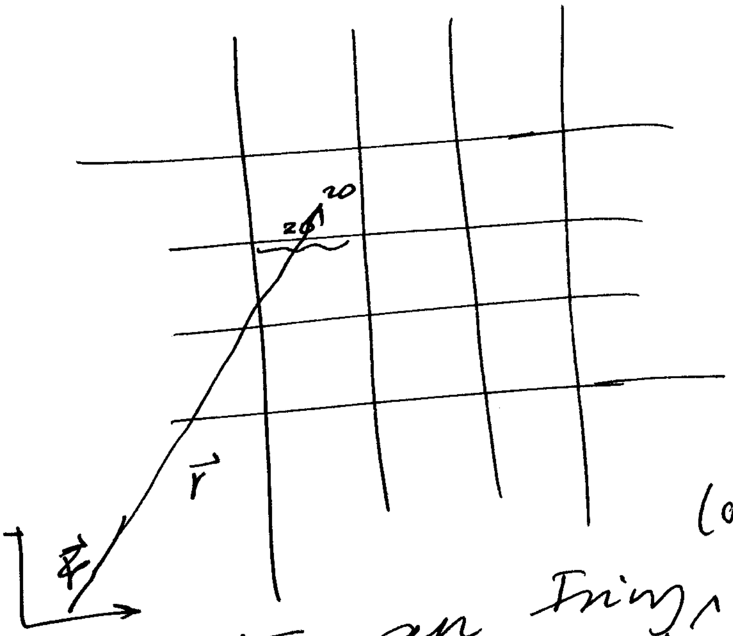
$$Z = \int \mathcal{D}[\vec{\phi}(\vec{r})] \, e^{-\beta H_{LG}[\vec{\phi}(\vec{r})]}$$

where $\mathcal{D}[\vec{\phi}(\vec{r})]$ is ~~the~~ denoting the functional integral. Several remarks are in order.

① Note that we have slyly changed our arena! Even if we had started with an Ising on ~~d~~ a d-cubic lattice with an order parameter we have ended up with an order parameter that depends on $\vec{r}$ in a continuous way. In other words we are ~~saying that~~ insisting that the arena is d-dimensional space, and our dofs are the order parameter fields $\vec{\phi}(\vec{r})$. What we now have is a <u>field theory</u>. LG approach therefore makes the stat mech problem equivalent to a field theory problem! And what is more, the field theory of elementary particles (standard model) is written ~~out~~ down in precisely same spirit as Landau-Ginzburg! Identify the fields and their symmetries (both global and internal) and write out the Lagrangian ~~de~~ density that respects these symmetries (which include local gauge symmetries)! ~~So one~~ In fact ~~thus~~ the LG approach has become a way (paradigm) of doing physics! One also sees the nice connection between equilibrium statmech and (quantum) field theory. While these may look rather high sounding, these connections are quite real and infact useful in solving problems!

② You could ask - but what is the connection of all this to ~~the~~ the microscopic - Ising model that we wrote down? In other words, can one "<u>derive</u>" starting from the Ising model, the LG Hamiltonian? Actually a discussion of this ~~point~~ turns out to be quite interesting!

The process of deriving the GL functional is by <u>coarse graing</u>: The idea is this:



Consider the an Ising (or O(n)) system which is very large. Dice it up into little chunks say made up of 20x20 squares. Now let $\vec{r}$ be the coordinate of the center of the ~~and~~ small square. Now define

$$\vec{\phi}(\vec{r}) = \frac{1}{N_s}\sum_i \vec{S}_i \qquad N_s = 20\times 20 \text{ (in our example)}.$$

Now

$$e^{-\beta H_{GL}[\vec{\phi}(\vec{r})]} \sim \sum_{S_i} e^{-\beta H(S_i)} \prod_r \delta\left(\vec{\phi}(\vec{r}) - \frac{1}{N}\sum_i \vec{S}_i\right)$$

~~This is the~~ This procedue is quite hard (impossible) to take through. Landau and Ginzburg argue that the <u>form</u> of H_{GL} is determined by symmetry and analyticity.

What this discussion brings out is a very crucial idea of ~~Ginzburg~~ Landau-Ginzburg theory -which is that the position ~~x~~ variable $\vec{r}$ is also coarse grained. In the example above suppose ~~x~~ I know $\vec{\phi}(\vec{r})$, I am not allowed to ask what is $\phi(\vec{r}+(10,10))$, because $\phi(\vec{r})$ is a coarse grained variable!

This idea can be stated most precisely in Fourier language what is fourier transform of $\vec{\phi}(\vec{r})$?

$$\vec{\phi}(\vec{r}) = \cancel{\sum_{\vec{k}}} \int_{|\vec{k}|<\Lambda} d^d\vec{k}\ \phi(\vec{k}).$$

where Λ is the *momentum or ultraviolet cut off*. In our stat mech problem this is the *length over which we have coarse grained*. In our example $\Lambda = \frac{1}{20}$. or of the order of $\Lambda \sim \frac{\pi}{a_0}$ where a_0 is the lattice spacing. Thus although we have written out the LG ~~Hamiltonian~~ in a simple form, there is this crucial idea ~~hid~~ hiding of ultraviolet cutoff that is underneath. We can now be more explicit by writing

$$f_L(t,u;\vec{\phi}(\vec{r})) = \frac{t}{2}|\vec{\phi}|^2 + \frac{u}{4}|\vec{\phi}(\vec{r})|^4$$

(note change of notation)

This page is added later, please bear with me. (Break in continuity).

Dimensional Analysis

$$\beta H_{LG}[\phi(\vec{r})] = \int d^d r \; \frac{K}{2}\left[\vec{\nabla}\vec{\phi}\right]^2 + \frac{t}{2}|\vec{\phi}|^2 + \frac{u}{4}|\vec{\phi}(\vec{r})|^4 - h(r)\phi(r)$$

L.H.S. is dimensionless.

$t \rightarrow$ dimensionless by choice, $[r] = L$ ↑ length dimension

$$[\phi(\vec{r})] = L^{-d/2}$$

$[K] = L^2$, and we define

$\xi_0^2 = K$ where ξ_0 is some (phenomenological) scale.

$$[u] = L^d.$$

$$[h(r)] = L^{-d/2}$$

$[u] \times L^d * (L^{-d/2})^4$

$[u] = L^d$

Note that this is not the only way to choose units. We will actually choose a slightly different system later on.

I will now do this a bit more carefully to get all the factors correctly since we will need this later.

Now, $$\vec{\phi}(\vec{r}) = \frac{1}{V} \sum_{|\vec{k}|<\Lambda}{}' \vec{\phi}(\vec{k})\, e^{i\vec{k}\cdot\vec{r}}.$$

~~Now~~ where V is the volume of our system. Note that since $\vec{\phi}(\vec{r})$ is real $\vec{\phi}^*(\vec{k}) = \vec{\phi}(-\vec{k})$

Now, $\int d^d r \; \frac{t}{2} \vec{\phi}(\vec{r})\cdot\vec{\phi}(\vec{r})$

$$= \frac{t}{2} \int d^d\vec{r} \; \frac{1}{V^2} \underbrace{\sum_{\vec{k}}{}' \sum_{\vec{k}'}{}'}_{|\vec{k}|,|k'|<\Lambda} e^{i(\vec{k}+\vec{k}')\cdot\vec{r}}\, \vec{\phi}(\vec{k})\cdot\vec{\phi}(\vec{k}')$$

$$= \frac{t}{2} \; \frac{1}{V} \sum_{\vec{k}<\Lambda}{}' \vec{\phi}(-\vec{k})\cdot\phi(\vec{k})$$

$$= \frac{t}{2} \int_{|\vec{k}|\leq\Lambda} \frac{d^d k}{2\pi^d} \; \underbrace{\vec{\phi}^*(\vec{k})\cdot\vec{\phi}(\vec{k})}_{\equiv |\vec{\phi}(\vec{k})|^2}$$

By a similar analysis we get

$$\int d^d r \; \frac{K}{2} |\vec{\nabla}\vec{\phi}|^2 = \frac{K}{2} \int_{|\vec{k}|<\Lambda} \frac{d^d k}{(2\pi)^d} |\vec{k}|^2 |\vec{\phi}(\vec{k})|^2$$

Let us look at the quartic term, we get

$$\frac{u}{4}\int d^d\vec r\,\frac{1}{V^4}\sum_{\vec k_1\vec k_2\vec k_3 k_4}{}' e^{i(\vec k_1+\vec k_2+\vec k_3+k_4)\cdot\vec r}\,\vec\phi_{m_1}(\vec k_1)\,\vec\phi_{m_2}(\vec k_2)\,\vec\phi_{m_2}(\vec k_3)\cdot\vec\phi_{m_4}(\vec k_4)$$

$$= \frac{u}{4}\int\prod\frac{d^d\vec k_i}{(2\pi)^d}\,\vec\phi_{m_1}(\vec k_1)\,\phi_{m_2}(\vec k_2)\,\phi_{m_3}(\vec k_3)\,\phi_{m_1}(\vec k_4)\,(2\pi)^d\delta(\vec k_1+\vec k_2+\vec k_3+\vec k_4)$$

The LG action therefor becomes

$\beta H_{LG}[\vec\phi(\vec k)]$ written in momentum space

$$= \int\frac{d^d k}{(2\pi)^d}\left(\frac{K}{2}|\vec k|^2+\frac{t}{2}\right)|\vec\phi(\vec k)|^2$$

$$+\ \frac{u}{4}\int\prod\frac{d^d\vec k_i}{(2\pi)^d}\,\phi_{m_1}(\vec k_1)\,\phi_{m_2}(\vec k_2)\,\phi_{m_2}(\vec k_3)\,\phi_{m_1}(\vec k_1)\,(2\pi)^d\delta(\vec k_1+\vec k_2+\vec k_3+\vec k_4)$$

and in real space this is (with a magnetic field)

$$\beta\mathcal{H}_{LG}[\vec\phi(x)] = \int d^d\vec r\ \frac{K}{2}|\vec\nabla\vec\phi|^2+\frac{t}{2}|\vec\phi(\vec r)|^2+\frac{u}{4}|\vec\phi(\vec r)|^4-\vec h(\vec r)\phi(\vec r)$$

Now we can write out the partition function as

$$Z = \int \mathcal{D}[\vec{\Phi}(x)] \; e^{-\beta H_{LG}[\vec{\Phi}(x)]}$$

or

$$Z = \int_{\substack{[\text{only half of} \\ \vec{k}\text{ space} \\ \text{owing to } \vec{\Phi}^*(\vec{k}) = \vec{\Phi}(-\vec{k})]}} \mathcal{D}[\vec{\Phi}^*(\vec{k}), \phi(\vec{k})] \; e^{-\beta H_{LG}[\vec{\Phi}(\vec{k})]}.$$

We are in a position to ask some physics questions. For example what is the relationship between $\vec{M}$ of the Landau theory (in particular the equilibrium value of $\vec{M}$) and $\vec{\Phi}(\vec{r})$. It should now be pretty clear that

$$\vec{M} = \langle \vec{\Phi}(\vec{r}) \rangle_{LG} = \frac{\int \mathcal{D}[\vec{\Phi}(\vec{r})] \, e^{-\beta H_{LG}[\vec{\Phi}(\vec{r})]} \, \vec{\Phi}(\vec{r})}{Z}$$

or

$$\vec{M} = \frac{1}{V} \langle \vec{\Phi}(\vec{k} = \vec{0}) \rangle$$

But now we are faced with a difficult problem! How do we evaluate the partition function? In other words How do we do the functional integral?

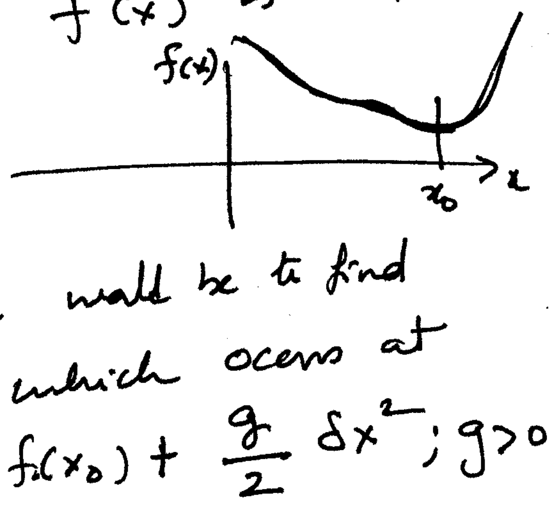
There is a whole generation of technical machinery that has been developed to do this kind of thing. We will certainly not be able to get into the details of that in our course. You may have to pick this up on your own or take some of the results for granted (but they will only be technical results... You can still follow the physics).

One of these techiques is called the saddle point analysis. Let us look at this techique via a simple example.

Suppose I want to evaluate

$$\int dx \, e^{-f(x)}$$

where $f(x)$ is a function that looks something like:



Then the saddle point value would be to find the minimum value of $f(x)$, which occurs at x_0. Then expand $f(x) = f(x_0) + \frac{g}{2}\delta x^2$; $g > 0$

$\delta x = x - x_0$

Now
$$\int dx \, e^{-f(x)} \simeq \int d(\delta x) \, e^{-\left(f(x_0) + \frac{g}{2}\delta x^2\right)}$$
$$= e^{-[f(x_0)]} \sqrt{\frac{2\pi}{g}}$$

$$= e^{-\left[f(x_0) + \frac{1}{2}\log\left(\frac{g}{2\pi}\right)\right]}$$

This is the idea. Now, we have to generalize this to functional integrals. The idea is the following: Need to find

$$Z = \int \mathcal{D}[\vec\phi]\, e^{-\beta\mathcal{H}_{LG}[\vec\phi]},$$

Find the saddle point function $\vec\Phi_0$; in other words find the field Φ_0 that extremises the functional $\beta\mathcal{H}_{LG}[\vec\Phi_0]$, This $\vec\Phi_0$ will satisfy an Euler-Lagrange equation related to the functional βH_{LG}. Now write

$$\vec\Phi = \vec\Phi_0 + \vec\psi \xrightarrow{}$$ fluctuation about the saddle point function $\vec\Phi_0$.

Now
$$\beta\mathcal{H}_{LG}[\vec\phi] = \beta\mathcal{H}_{LG}[\vec\Phi_0] + \beta H^{(2)}_{LG}[\vec\psi]$$

where $\beta\mathcal{H}^{(2)}_{LG}[\vec\psi]$ is a quadratic functional of $\vec\eta$. This will then reduce

$$Z \simeq e^{-\beta\mathcal{H}_{LG}[\vec\Phi_0]} \int \mathcal{D}[\vec\psi]\, e^{-\underbrace{\beta\mathcal{H}^{(2)}_{LG}[\vec\psi]}_{\text{quaratic functional of } \vec\psi}}$$

Now it turns out that ~~th~~ that functional integrals of this type (Gaussian functional integrals) can ~~at~~ actually be done! Thus the saddle point approch allows us to do calculations. How ever the key thing is that fluctuations about the saddle point $\vec{\Phi}_0$ has to be "small" for this appoach to be reliable. In other words, ~~the~~ ~~[illegible]~~ $\vec{\Phi}_0$ must correspond to a deep minimum of βH_{LG}. Let us now see this method in action.

To determine $\vec{\Phi}_0$, for our LG functional ~~[illegible]~~ write out its Euler Lagrange equation

$$\vec{\nabla}\cdot\left(\frac{\delta \beta H_{LG}}{\partial \vec{\nabla}\vec{\phi}}\right) - \frac{\partial \beta H_{LG}}{\partial \vec{\Phi}} = 0$$

$$\Rightarrow \quad K\nabla^2\vec{\Phi} - \frac{\partial f_L(t,u;\vec{\Phi})}{\partial \vec{\Phi}} = \vec{h} \text{ (uniform)}$$

Now it is possible that this equation has many solutions. (depending on boundary condition at infinity). We can choose a uniform solution $\vec{\Phi}_0(\vec{r}) = \vec{\Phi}_0$ (constant vector) and this then satisfies the equation

$$t\vec{\Phi} + u|\vec{\Phi}|^2\vec{\Phi} = \vec{h}$$

$$\Rightarrow \quad \vec{\Phi}_0 = \left\{\begin{array}{ll} \vec{0} & t>0 \\ |\vec{\Phi}_0| = \sqrt{\frac{-t}{u}} & t<0 \end{array}\right\} \text{ for } \vec{h}=0$$

~~[illegible]~~

For $t<0$, the direction of $\vec{\Phi}_0$ (for $O(n)$ $n\geq 2$) is not determined by theory. This direction is spontaneously chosen by the system. We ~~chan~~ can choose the direction chosen to be 1 direction

$$(\vec{\Phi}_0)_{m=1} = \phi_0 = \sqrt{\frac{-t}{u}} \quad (t<0$$

$$(\vec{\Phi}_0)_{m\geq 2} = 0 \qquad \begin{array}{l} m \geq 2 \ldots n \\ \alpha = 2 \ldots n \end{array}$$

We also set $(\vec{h})_{m=1} = h$ and $(\vec{h})_\alpha = 0$.

For $h=0$, ϕ_0 satisfies

$$\boxed{t\phi_0 + u\phi_0^3 = h}$$

Now

$$\phi_m = \begin{cases} \phi_0 + \psi_1 & m=1 \\ \psi_\alpha & \alpha = m \geq 2 \end{cases}$$

Now

$$\beta H_{GL}[\vec{\phi}] = \beta H_{GL}[\vec{\Phi}_0] + \int d^d r \left[\frac{K}{2} |\vec{\nabla}\vec{\psi}|^2 + \frac{t}{2}|\psi|^2 + \frac{u}{4}\left[6\phi_0^2\psi_1^2 + 2\phi_0^2 \underbrace{\psi_\alpha\psi_\alpha}_{\text{Repeated summed.}}\right]\right.$$

to quadratic order in the ψ-fields.

Now we can rearrange this a bit to see that

$$\beta H_{GL}^{\cancel{(2)}}[\vec{\phi}] = \beta H_{GL}[\vec{\phi}_0] + \beta H_{GL}^{(2)}[\vec{\psi}]$$

where

$$\beta H_{GL}^{(2)}[\vec{\psi}] = \int d^d r \left[\frac{K}{2}(\vec{\nabla}\psi_1)^2 + \left[\frac{t}{2} + \frac{3u}{\cancel{4}}\phi_0^2\right]\psi_1^2 \right. + \sum_{\alpha}{}' \int d^d r \; \frac{K}{2}(\vec{\nabla}\psi_\alpha)^2 + \frac{1}{2}(t+u\phi_0^2)\psi_\alpha^2$$

noting that ~~to~~ noting that f $t<0$

$t+u\phi_0^2 = \frac{h}{\phi_0}$, we get

$$\beta \mathcal{H}_{GL}^{(2)}[\vec{\psi}] = \int d^d r \left[\frac{K}{2}(\vec{\nabla}\psi_1) + \left[\frac{h}{2\phi_0} + u\phi_0^2\right]\psi_1^2 + \sum_{\alpha}{}' \frac{K}{2}(\vec{\nabla}\psi_\alpha)^2 + \frac{h}{2\phi_0}\psi_\alpha^2 \right].$$

To calculate the partition function, (and the free energy) we need

$$Z = e^{-\beta \mathcal{H}_{GL}[\vec{\phi}_0]} \int \mathcal{D}[\vec{\psi}]\, e^{-\beta \mathcal{H}_{GL}^{(2)}[\vec{\psi}]}$$

Actually we can do this type ~~fr~~ of functional integral. Go to the momentum space action $\bar{d}^dk = \frac{d^dk}{(2\pi)^d}$

$$\beta\mathcal{H}_{GL}^{(2)}\left[\vec{\phi}(\vec{k})\right] = \int_{|\vec{k}|<\Lambda} \bar{d}^dk \left[\frac{K}{2}k^2 + \frac{h}{2\phi_0} + u\phi_0^2\right]|\psi_1(\vec{k})|^2$$

$$+ \sum_\alpha{}' \int_{|\vec{k}|<\Lambda} \bar{d}^dk \left[\frac{K}{2}k^2 + \frac{h}{2\phi_0}\right]|\psi_\alpha(k)|^2$$

and

$$Z = e^{-V\beta\mathcal{H}_{GL}[\vec{\phi}_0]} \int \prod_{\vec{k}} \mathcal{D}\left[\vec{\phi}^*(\vec{k})\,\vec{\phi}(\vec{k})\right] e^{-\beta\mathcal{H}_2^{GL}}$$

only for half the $\vec{k}$ space

$$Z = e^{-\beta V H_{GL}[\vec{\Phi}_0]} \sqrt{\prod_{|\vec{k}|<\Lambda}{}' \left(\frac{\pi}{\left(K_1k^2 + \frac{h}{\phi_0} + 2u\phi_0^2\right)}\right) \prod_\alpha \frac{1}{\left(K_1k^2 + \frac{h}{\phi_0}\right)}}$$

$$\Rightarrow \beta F = V\beta H_{GL}[\vec{\phi}_0] - \sum_{|\vec{k}|<\Lambda}{}' \ln\left(\frac{\pi}{K_1k^2 + \frac{h}{\phi_0} + 2u\phi_0^2}\right)$$

$$+ (n-1)\sum{}' \frac{\pi}{\left(K_1k^2 + \frac{h}{\phi_0}\right)}$$

Now set the external field ~~h→~~ $h \to 0$.

$t > 0 \qquad \frac{h}{\phi_0} \simeq 1/t \qquad \phi_0^2 \to 0 \quad h \to 0$

$$\frac{\beta F}{V} = \beta H_{GL}[\vec{\phi}_0] - \frac{1}{2}\left[\sum_{|\vec{k}|<\Lambda}' \ln\left(\frac{\pi}{Kk^2+t}\right) + (n-1) \sum_{|\vec{k}|<\Lambda}' \ln\left(\frac{\pi}{Kk^2+t}\right) \right]$$

$t \to 0$

Then $$\frac{\beta F}{V} = \beta H_{GL}[\vec{\phi}_0] - \frac{n}{2(2\pi)^d} \int_{|k|<\Lambda} d^dk \, \ln\left(\frac{\pi}{Kk^2+t}\right)$$

(very near T_c)

area of unit sphere in d-dims.

$$\& \; C_v = -\frac{\partial^2 \beta F}{\partial t^2}$$

$$= 0 - \frac{n S_d}{2(2\pi)^d} \int_0^\Lambda dk \, k^{d-1} \frac{1}{(Kk^2+t)^2}$$

$$C_v = \frac{n S_d}{2(2\pi)^d} \int_0^\Lambda dk \, k^{d-1} \frac{1}{(Kk^2+t)^2}$$

define $$y = \sqrt{\frac{K}{t}}\, k$$

Then $$C_v = \frac{n S_d}{2(2\pi)^d} \left(\sqrt{\frac{t}{K}}\right)^d \frac{1}{t^2} \int_0^{\sqrt{\frac{K}{t}}\Lambda} dy \, \frac{y^{d-1}}{(Ky^2+1)^2}$$

$$= \frac{n S_d}{2(2\pi)^d} \frac{1}{K^{d/2}} \, t^{d/2-2} \, f(\Lambda)$$

For $t<0$ $\lim_{h\to 0} \frac{h}{\phi_0} = 0$! since ϕ_0 already has a broken symmetry value.

Thus for $t<0$

$$\frac{\beta F}{V} = \beta H_{GL}[\vec{\Phi}_0] - \frac{1}{2}\sum_{|\vec{k}|<\Lambda}' \ln \frac{\pi}{(Kk^2+2|t|)} - \frac{(n-1)}{2}\sum_{|\vec{k}|<\Lambda}' \ln \frac{\pi}{(Kk^2)}$$

We thus see that (singular part only)

$$C_V \sim -\frac{S_d}{2(2\pi)^d}\int_0^{\Lambda} dk \, \frac{k^{d-1}}{(Kk^2+2|t|)^2}$$

$$C_V \sim \frac{S_d}{2(2\pi)^d}\left(\frac{2|t|}{K}\right)^{d/2}\frac{1}{(2|t|)^2} f(\Lambda)$$

Final result ~~α~~ $C_V \sim C_\pm^{\alpha} |t|^{-\alpha}$

$$\alpha = \frac{4-d}{2} \qquad \Bigg| \qquad \frac{C_+^{\alpha}}{C_-^{\alpha}} = n \, 2^{\frac{4-d}{2}}$$

↑ exponent

↑ Amplitude ratio of specific heat

We see that exponent is independent ~~of~~ of n and dependent on d. Amplitude ratio is dependent on n and d.

Now with in the gaussian fluctions theory the order parameter is

$$\vec{M} = \frac{\int \mathcal{D}[\vec{\phi}]\, e^{-\beta \mathcal{H}_{GL}} \vec{\phi}(\vec{x})}{Z}$$

$$\simeq \frac{\int \mathcal{D}[\vec{\psi}]\, e^{-(\beta \mathcal{H}_{GL}[\vec{\phi}_0] + \beta \mathcal{H}^{(2)}_{GL}[\vec{\psi}])} (\vec{\phi}_0 + \vec{\psi}(\vec{x}))}{Z_{gaussian}}$$

$$= \vec{\phi}_0 \; !$$

We see immediatley that at $\vec{h} = \vec{0}$.

$$\vec{M} = \begin{cases} \cancel{\phi}\ 0 & t > \\ \sqrt{\frac{-t}{u}}\ \vec{e}_1 & t < 0 \end{cases}$$

and thus $\beta = 1/2$

By the same token

$$\chi = \begin{cases} \frac{1}{2t} & t > 0 \\ \frac{1}{2|t|} & t < 0 \end{cases}$$

and hence $\gamma = -1$ $\quad \frac{C_+^{\gamma}}{C_-^{\gamma}} = 2$

and finally at $t=0$,

$$M = h^{1/3} \quad \text{and hence } \delta = 3.$$

We can collect all these results.

	Results of LG at Gaussian Level	
	Exponent	Ratio
$C_V =$	$\alpha = \frac{4-d}{2}$	$\frac{C_+^{\alpha}}{C_-^{\alpha}} = n\, 2^{\frac{4-d}{2}}$
M	$\beta = 1/2$	—
χ	$\gamma = 1$	$\frac{C_+^{\chi}}{C_-^{\chi}} = 2$
$M\,(t=0)$	$\delta = 3$	—

Everything Unexcept The specific characteristics changed for $d < 4$! We see that there is a divergent specific heat at lower dimensions. What is the cause of this? To understand this we need to look more carefully at what our calculations mean. For $t \geq 0$, we find that the Gaussian action (saddle point + quadratic fluctuations)

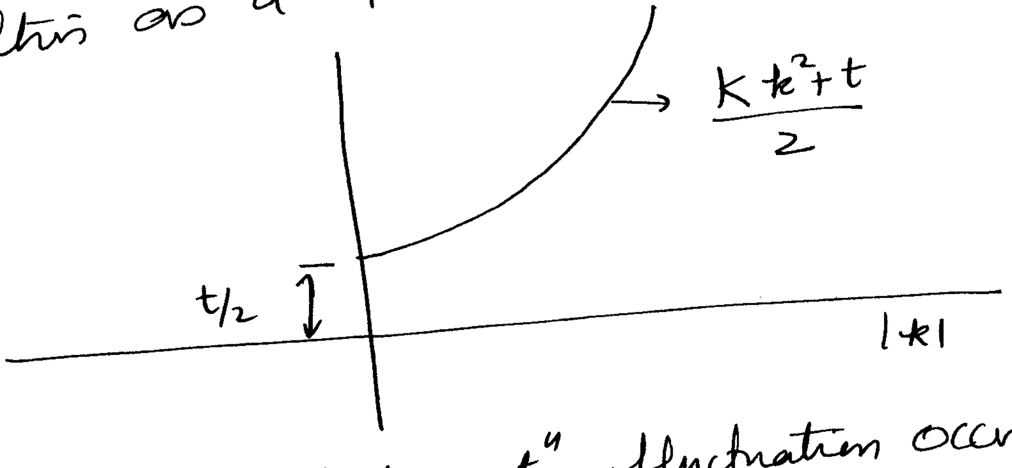
For $t > 0$ $(h = 0)$

$$\beta H^{gam}_{GL}[\vec{\phi}] = \beta H_{GL}[\vec{\phi}_0] + \sum_m{}' \int_{|\vec{k}|<\Lambda} d^d k \left(\frac{K k^2 + t}{2}\right) \phi^*_m(k)\, \psi_m(\vec{k})$$

~~$\phi^*(k)$~~) $\psi_m(k)$

We now see that the "small oscillation" fluctuations are the the

The energy cost of having a fluctuation of wavevector k with amplitude $\psi_m(\vec{k})$ is $\left(\frac{K k^2 + t}{2}\right)|\psi_m(\vec{k})|^2$. The quantity $\left(\frac{K k^2 + t}{2}\right)$ is the "stiffness". Let us plot this as a function of $|k|$



We see that the "cheapest" fluctuation occurs at $k=0$, the long wavelength fluctuations. This has an energy cost $\sim t > 0$. Thus these fluctuations are "not free", they cost finite energy and hence are <u>gapped</u>.

In fact all the n modes for ~~a~~ every ~~given case~~ $\vec{k}$ are gapped. In this sense the system ~~[illegible]~~ has fully gapped fluctuations. But notice that the gap of all these modes is becoming smaller as $t \to 0$. They can therefore get thermally excited (since there is a lot of entropy gain). As $t \to 0$, many many modes with "small" k are excited, ~~hence~~ the large specific heat. ~~Why~~ what happens as a function of dimension.

We see that the density of states of the modes

$$g(\mathcal{E})\, d\mathcal{E} = c_d\, k^{d-1}\, dk$$

$$\mathcal{E}(k) = K k^2 + \mathcal{E}_0 \qquad \begin{cases} \mathcal{E}_0 = t & t>0 \\ \phantom{\mathcal{E}_0} = 2|t| & t<0 \end{cases}$$

($\mathcal{E}_0$ is the excitation gap).

$$g(\mathcal{E})\, d\mathcal{E} = \frac{c_d}{2} \left(\frac{\mathcal{E} - \mathcal{E}_0}{K} \right)^{d-2} d\mathcal{E}$$

We can rewrite the specific heat result ~~as~~

$$C_V = \frac{[1+\theta(t)(n-1)]}{4}\int_{\varepsilon_0}^{K\Lambda^2+\varepsilon_0} d\varepsilon \, \frac{\left(\frac{\varepsilon-\varepsilon_0}{K}\right)^{\frac{d-2}{2}}}{\varepsilon^2}$$

Let us study the behavior for infrared and ultra violet Energies. In the infra reed, the integrand goes as

$$\frac{1}{\varepsilon_0^2}\left(\frac{\varepsilon-\varepsilon_0}{K}\right)^{\frac{d-2}{2}},$$

~~this~~ this ~~for~~ this ~~term~~ the infrared makes a significant contribution when d is smaller.

On the other hand, the ~~to~~ ultraviolet looks like $\varepsilon^{\frac{d-4}{2}}$ which gives a finite results for $d < 4$, but makes large contributions starting $d \geq 4$. For $d = 4$

$$C_V \sim \int_{\varepsilon_0}^{K\Lambda^2+\varepsilon_0}\left(\frac{\varepsilon-\varepsilon_0}{\varepsilon^2}\right) \sim \ln(|t|)$$

For $d > 4$

and in $d=4$ there is a log divergence. and ~~this~~ ~~term~~ there is ~~a~~ no divergence and specific heat for $t=0$ would go to zero as $(-t)^{\frac{4-d}{2}}$

and thus the jump discontinuity predicted by Landau would be the correct answer. ~~Fa~~ This is because the infrared density of states is smaller for $d > 5$, and the entropy gain of the fluctuations is far less effective compared to their energetic cause. The gaussian theory therefore ~~does~~ is likely to hold for $d \geq 5$. For $d < 4$, the story is ~~atot~~ altogether different. One can see that there are lots of ~~d~~ excitations in the infrared and this causes large fluctuations to occur so as to produce a divergent specific heat. The key to this divergence, again, are the low energy (infrared excitations). In $d = 4$ all ~~ext~~ excitations ~~cont~~ (at all wave lengths) contribute equally to produce a log divergence

We thus see that for the $O(n)$ model $d = 4$ is a crucial dimension. For $d > 4$, fluctuations at the Gaussian level do not qualitatively change the physics, and the transition as given by the Landau theory (or mean field theory)

For O(n) model, in d<4, fluctuations are important enough to give a divergence of the specific heat. This will make us wonder if the order predicted by the saddle point approach is a good one, i.e., is the order stable to fluctuations? Let us calculate how big the fluctuations are. A quantity that will be of repeated interest ~~th~~ is the fluctuation correlation function

$$\langle \psi_m(\vec r)\, \psi_{m'}(\vec r')\rangle$$

$$= \frac{1}{V^2}\sum_{k,k'} e^{i(\vec k\cdot\vec r+\vec k'\cdot\vec r')}\,\langle \Phi_m(\vec k)\,\psi_{m'}(k)\rangle$$

What is $\langle \psi_m(\vec k)\,\psi_{m'}(\vec k')\rangle$? The exact answer is hard to calculate. Within the saddle point approximation ~~it~~ it is easy to see that

$$\langle \phi_m(\vec k)\,\phi_{m'}(\vec k')\rangle = \underset{\substack{\uparrow\\ \text{Volme}}}{V}\,\delta_{\vec k',-\vec k}\,\delta_{m,m'} \times G_m(k)$$

where

$$G_m(k) = \begin{cases} \dfrac{1}{K k^2 + t} & t > 0 \end{cases}$$

$$= \begin{cases} m=1 \quad \dfrac{1}{K k^2 + 2|t|} \\ m=\alpha \\ =2..n \quad \dfrac{1}{K k^2} \end{cases} \quad t < 0$$

Let us calculate first for $t > 0$

$$\langle \psi_m(\vec r)\, \psi_m(\vec r') \rangle$$

$$= \frac{1}{V} \sum_{\vec k} e^{i\vec k\cdot(\vec r - \vec r')} \frac{1}{K k^2 + t}$$

$\boxed{\text{Can choose } \vec r' = 0}$

$$= \frac{1}{(2\pi)^d} \int d^d \vec k \; e^{i\vec k\cdot\vec r} \frac{1}{K_1 k^2 + t}$$

$$= \frac{1}{K_1} \frac{1}{(2\pi)^d} \int d^d k \; e^{i\vec k\cdot\vec r} \frac{1}{k^2 + \frac{t}{K_1}}$$

Note $\frac{t}{K_1}$ has units of L^{-2} (inverse length square. Call $\frac{t}{K_1} = \xi^{-2}$.

Define

$$G^d(\vec r) = \frac{1}{(2\pi)^d}\int d^dk \quad e^{i\vec k\cdot\vec r}\,\frac{1}{k^2+\xi^{-2}}$$

This $G^d(r)$ is simply the Green's function of the Helmhotz equation in d-dimension

$$(-\nabla^2 + \xi^{-2})\, G^d(\vec r) = \delta(\vec r)$$

In fourier space

$$(k^2+\xi^{-2})\,\underbrace{G^d(\vec k)}_{\substack{+\\ = \;.}} = 1.$$

By symmetry $G^d(\vec r)$ depends only on $|\vec r|$.

In d-dimension ∇^2 ~~took~~ is

$$\nabla^2 = \frac{1}{r^{d-1}}\frac{\partial}{\partial r}\left(r^{d-1}\frac{\partial}{\partial r}\right) + \frac{1}{r^d}\hat L_d^2$$

$\hat L_d^2$ is the angular momentum operation in d-dimension. ($\hat L_d = 0$ in $d=1$)

We are looking at the spherically symmetric case and hence the "s"-wave state which has $\hat L_d$ eigenvalue $=0$.

We thus see that $G^d(r)$ satisfies the relation

$$-\left[\frac{d^2G^d}{dr^2} + \frac{d-1}{r}\frac{dG^d}{dr}\right] + \xi^{-2}G^d = 0 \quad (r>0)$$

Now, this equation has an exact solution in terms of Bessel functions. We will instead work out the behavior of $G^d(r)$ for $r \ll \xi$ and $r \gg \xi$. For this purpose we make the substitution

$$G^d(r) = \frac{H^d(r)}{r^{\frac{(d-1)}{2}}}$$

Then the differentail equation becomes

$$-\frac{d^2H^d}{dr^2} + \frac{(d-1)(d-3)}{4}\frac{H^d}{r^2} + \xi^{-2}H^d = 0$$

Now in the $r \ll \xi$ limit we can drop the last term to obtain

$$-\frac{d^2H^d}{dr^2} + \frac{(d-1)(d-3)}{4}\frac{H^d}{r^2} = 0$$

to obtain

$$H^d(r) = \begin{cases} r^{\frac{(d-1)}{2}} & \text{or} \\ r^{\frac{(3-d)}{2}} \end{cases}$$

the 1st solution is not singular enough so we ignore it what happes for $r \ll \xi$ is

$$G^d(r) \approx \frac{r^{\frac{3-d}{2}}}{r^{\frac{(d-1)}{2}}} = \frac{1}{r^{d-2}} .$$

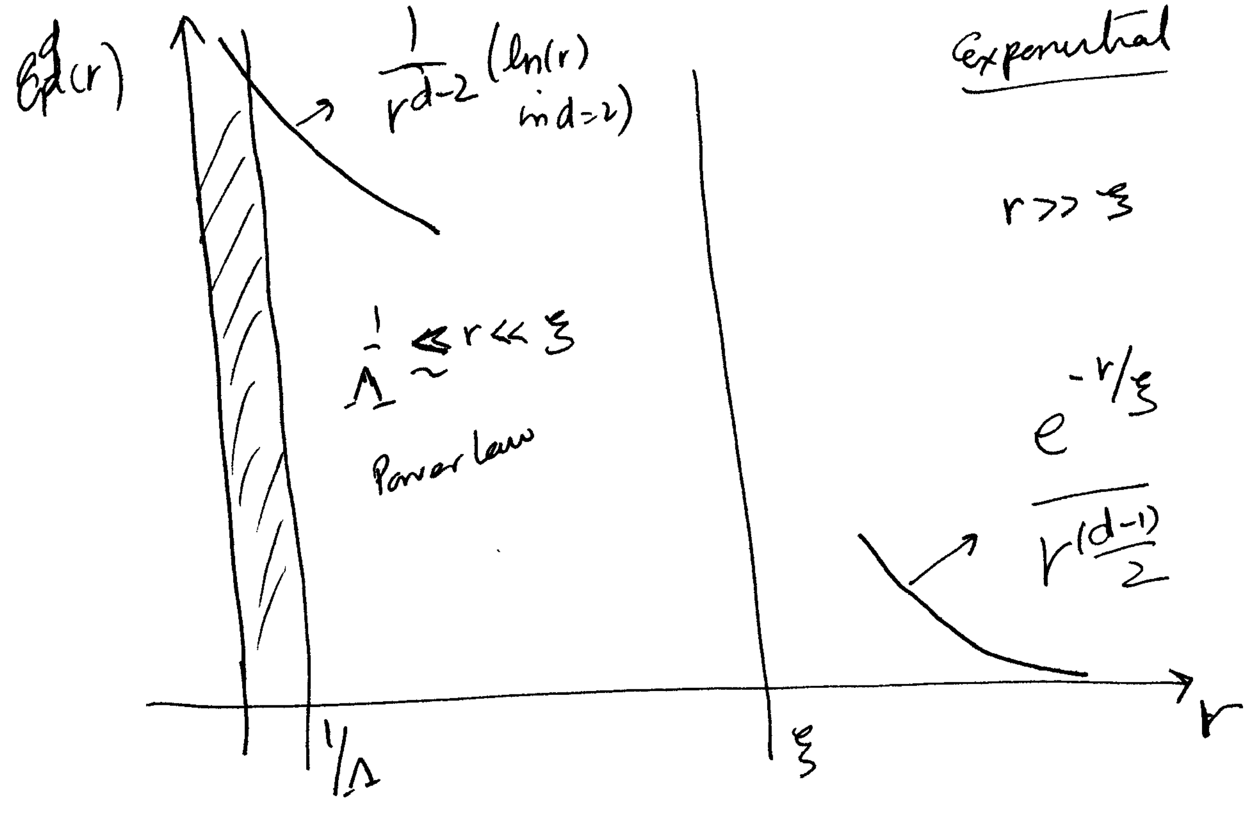
[for $d=2$ this will be $\ln(r)$]

We see that $G^d(r)$ has power law behaviour. For $\xi \ll r$, we see that the 2nd term in the diffential equation can be dropped and we get

$$H^d(r) \approx e^{-r/\xi}$$

or $$G^d(r) \approx \frac{e^{-r/\xi}}{r^{\frac{(d-1)}{2}}}$$

We see that $G^d(r)$ has the following behaviour



Thus we see that ~~d~~ at distances smaller than the correlation length there is a dimension dependent *power law correlation*, while for $r \gg \xi$

$$G^d(r) \sim \frac{e^{-r/\xi}}{r^{\frac{(d-1)}{2}}}.$$

~~Note that~~ ξ is called the correlation length

$$\xi = \sqrt{\frac{K}{t}} \qquad t > 0$$

and *diverges* as $t \to 0$. By definition $\xi = |t|^{-\nu}$ and $\nu = 1/2$ $t > 0$

For $t > 0$

$$\langle \psi_m(\vec{r})\, \psi_{m'}(0) \rangle = \delta_{mm'}\, G_0^d(r)$$

~~[illegible] for $r \to 0$~~

note that $r \to 0$. $\langle \psi_m(\vec{r})\, \psi_{m'}(0) \rangle$ is blowing up. Note that there is an important point here. Recall that $\vec{r}$ actually stands for $(\vec{r} - \vec{r}\,')$. Also recall that the theory has a cut off and $|\vec{r}| \gtrsim 1/\Lambda$. So one cannot naively take $r \to 0$ limit since we will be asking too much

Let us understand this better (the $r \to 0$ behaviour).

$$G^d(|\vec{r}|) = \int \frac{d^d k}{(2\pi)^d} \frac{e^{i\vec{k}\cdot\vec{r}}}{K k^2 + t} \qquad (t>0)$$

define a dimensionless vector $\vec{y} = r\vec{k}$

Then

$$G^d(\vec{r}) = \frac{1}{K\, r^{d-2}} \int \frac{d^d \vec{y}}{(2\pi)^d} \frac{e^{i\vec{y}\cdot\hat{r}}}{y^2 + (\frac{r}{\xi})^{+2}}$$

$$= \frac{1}{K\, r^{d-2}} \int_0^{\Lambda r} \frac{dy\, y^{d-1}}{(2\pi)^d} \int d\Omega_d \frac{e^{i\vec{y}\cdot\hat{r}}}{y^2 + (\frac{r}{\xi})^{+2}}$$

Note that since $r \gtrsim \frac{1}{\Lambda}$ the upper limit of y integral is bounded to 1. When we see that the what happens near $y \sim 0$. integrand (in $d \gtrsim 2$) is well behaved, In other words the infrared do not create problems.

Armed with this understanding let us go to the $t<0$, the ordered phase. Here, as we have seen before, there are two modes. The longitudinal mode $\psi_1(\vec r)$ which is gapped, and if $n>1$, the transverse modes $\psi_\alpha(\vec r)$ $(\alpha = 2 \ldots n)$. The longitudinal modes are gapped with a gap of $2|t|$ while the transverse modes are gapless.

Thus we see that

$$\langle \psi_1(\vec k)\, \psi_1(\vec k')\rangle \sim \frac{1}{K_1 k^2 + 2|t|}\,\delta_{\vec k,\vec k'}$$

$$\langle \psi_\alpha(\vec k)\, \psi_\alpha(\vec k')\rangle \sim \frac{1}{K k^2}\,\delta_{-\vec k,\vec k}$$

~~Thus we~~ Now define $\xi = \sqrt{\frac{K}{2|t|}}$ the correlation length. We see that

$$\xi \sim |t|^{-\nu} \Rightarrow \nu = 1/2. \quad (t<0)$$

Now

$$\langle \psi_1(\vec r)\, \psi_r(0)\rangle = \frac{1}{K_1}\frac{1}{(2\pi)^d}\int d^dk\, \frac{e^{i\vec k\cdot\vec r}}{k^2 + \xi^{-2}}$$

and behave exactly as before

$$\langle \psi_1(\vec r)\, \psi_1(0)\rangle = \begin{cases} \dfrac{1}{r^{d-2}} & \dfrac{1}{\Lambda} \lesssim r \ll \xi \\ \dfrac{e^{-r/\xi}}{r^{\frac{(d-1)}{2}}} & r \gg \xi \end{cases}$$

Now the fluctuations of the transeverse modes

$$\langle \psi_\alpha(\vec{r})\, \psi_\alpha(0) \rangle = \frac{1}{K_1'} \frac{1}{(2\pi)^d} \int d^d k \, \frac{e^{i\vec{k}\cdot\vec{r}}}{k^2}.$$

$$= \frac{1}{K_1'} \frac{1}{(2\pi)^d} \int \frac{dk\, k^{d-1}}{k^2} \int d\vec{\Omega}_d \, e^{ikr\cos\theta}$$

We now see that there is a peculiar problem in $d < 3$, i.e, starting from $d=2$; In $d=2$, we have

$$\langle \psi_\alpha(\vec{r})\, \psi_\alpha(0) \rangle = \frac{1}{K_1'} \frac{1}{(2\pi)^2} \int_0^\Lambda \frac{dk\, k}{k^2} \underbrace{\int d\theta\, e^{ikr\cos\theta}}$$

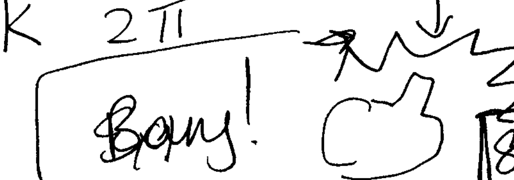
$$= \frac{1}{K_1'} \frac{1}{2\pi} \int_0^\Lambda \frac{dk\, k}{k^2} J_0(kr)$$

In the limit of $r \to 0$ $J_0(kr) \to 1$

Hence
$$\langle \psi_\alpha(\vec{r})\, \psi_0(0) \rangle \overset{r\to 0}{=} \frac{1}{K_1'} \frac{1}{2\pi} \int_0^\Lambda \frac{dk}{k}$$

$$\frac{1}{K} \frac{1}{2\pi} (\ln(\Lambda) - \ln(0))$$

$$= \infty!$$



We see that the the infinity is brought about by the gapless fluctuations, i.e, the infinity comes to an infrared divergence.

What we have seen is that if $n>1$, i.e, we have a continuous symmetry, the for $d \le 2$ we cannot have an ordered state, since the gapless excitations proliferate (large density of states in the infrared) and destroy the order. This is called the Mermin-Wagner Theorem[7].

Let us also discuss the excitations for $t<0$, a bit more carefully. There are two types (for $n>1$). Logitudinal excitations, which is gapped with a density of states

$$g_h(\varepsilon) = \frac{c_d}{2}\left(\frac{\varepsilon-\varepsilon_0}{K}\right)^{\frac{d-2}{2}} \qquad \varepsilon_0 = 2|t|.$$

This is called the Higgs mode. (not yet the Higgs boson).

The transeverse modes (there will be n-1 ~~as~~ of them for O(n) symmetry), are gapless. These are called goldstone modes. They have a density of states that goes as (for each mode)

$$g_g(\varepsilon) = \frac{c_d}{2}\left(\frac{\varepsilon}{K}\right)^{\frac{d-2}{2}}.$$

It is the proliferation of the Goldstone modes that destroys order in $d \le 2$. when the symmetry is continuous.

It turns out that the presence of such transverse gapless excitations is a general feature of broken symmetry states when the interactions in the system are short ranged (in real space). This usually goes under the name of "Goldstone theorem": stated once more; Breaking of a continuous symmetry ~~withe~~ (in system with short range interactions) will result in transverse excitations that are gapless. We can state the Mermin Wagner theorem ~~as~~ again: For $d \le 2$, the proliferation of goldstone modes destroy the longrange order. ~~&~~ In other words one cannot have a system with ~~a~~ broken continuous symmetry at finite temperatures in $d \leq 2$. [There are analogous results for quantum systems].

We are now in a position to summarize our results. For the $O(n)$ model, ~~that~~ the Gaussian fluctuation Landau Ginzburg theory, gives this that for a given n there are two ~~dimension~~ spatial dimensions $d_c^u(n)$, ~~and~~ the uppercritical dimension and $d_c^l(n)$ the lower critical dimension

n	$d_c^l(n)$	$d_c^u(n)$
1	–	4
≥ 2	2	4

For $n \geq 2$ no ordered state for $d \leq 2$. For $d > d_c^u(n)$, gaussian fluctuations theory is good.

According to Gaussian fluctuation theory what are the correlations at $t=0$, i.e, at the critical point?

$$\langle \psi_m(\vec r)\, \psi_m(0)\rangle \sim \int d^dk \, \frac{e^{i\vec k \vec r}}{k^2} \sim \frac{1}{r^{d-2}} .$$

There there are power law correlations at all ~~scales~~ length scales! This is also consistent with our earlier result for $t \neq 0$ since at $t=0$, $\xi \to \infty$. The correlation length diverges, there is no gap in the system, and therefore there is no characteristic scale in the system (really?)! This is a prediction of the LG theory within the Gaussian approximation. The divergence of the correlation length ξ is

$$\xi \simeq C^\nu_\pm \xi_0 |t|^{-\nu} \qquad \nu = 1/2$$

$$\frac{C_+^\nu}{C_-^\nu} = \sqrt{2} .$$

$$\xi_0 = \sqrt{K}$$

note none of this depends on d or n.

~~Nea~~ Now we must address another issue that we had encountered on page 46 How come there is a divergence in the specific heat of ^{4}He superfluid transition, while for the superconducting transition of Al, we obtain results consistent with the Gaussian fluction theory while both are in $d=3$! We know that for $d < d_c^u(n)$ the specic heat diverges as $(t<0)$ as

$$C_V^{\text{Fluctions}} = \frac{C_d}{2} \frac{2^{\frac{d-4}{2}} \, |t|^{\frac{d-4}{2}} \times (M_d)}{K^{d/2}}$$

$$M_d = \int_0^\infty dy \, \frac{y^{d-1}}{(y^2+1)^2}$$ (absorb ~~M~~ M_d and other constants into C_d).

$$C_V^{\text{Fluctions}} = \frac{C_d}{\cancel{K^{d/2}}} |t|^{\frac{d-4}{2}} \xi_0^{-d}$$

Now ask at what value of $|t|$ does this compare with the result of Landau theory?

$$C_V^{\text{Landau}} = \frac{1}{2u} \quad \left[\text{note dimen of } u = L^d\right]$$

~~How does~~ Set $C_V^{\text{Landau}} = C_V^{\text{fluction}}$

[illegible]

and obtain a temperature ~~t~~ t_G (Ginzburg temperature)

$$t_G = \left(\frac{\xi_0^d \, C_V^{\text{Landau}}}{C_d}\right)^{\frac{2}{d-4}}.$$

ξ_0 C_V^{Landau} is the specific heat which is crudely a measure of number of microscopic degrees of freedom per unit volume involved in the ordering process. ξ_0 is a microscopic scale (~~d~~ $C_d \sim 1$ drop)

$$t_G = \frac{1}{\left(\xi_0^d \, C_V^{\text{Landau}}\right)^{\frac{2}{4-d}}}$$

~~We see that~~ The argument about C_V^{Landau} can be obtained by noting the u has dimens. of L^{-d}. Now let us choose a scale a (some atomic scale). In typical transitions there is one μ-scopic degree of freedom per $1/a^d$ volume. Thus

$$C_V^{\text{Landau}} \sim \frac{1}{a^d}$$

$$t_G = \frac{1}{\left(\frac{\xi_0}{a}\right)^{\frac{2d}{4-d}}}$$

The key question is what is $\frac{\xi_0}{a}$? For ^4He $\xi_0 \sim 2\text{-}3\,a$, therefore

$$t_G = \frac{1}{2^3} \simeq 0.1 .$$ this is large.

(ξ_0 for ^4He is the thermal deBroglie wavelength

What happens in a superconductor? ξ_0 there is like the coherence length which is about 10000 Å (order of a micron). Thus

$$\frac{\xi_0}{a} \sim 10^3 \quad \text{and}$$

$$t_G = \frac{1}{(10^3)^6} \sim 10^{-18} \; !$$

Thus we see singular behaviour in specific heat only if we go

$$t \leq 10^{-18}$$

of T_c in usual superconductors! This is hard (impossible) to achieve, and we just see a jump in specific heat. This criterion is called the Ginzburg criterion which says that one should be within t_G of the transition to see the effect of fluctuations in specific heat. (other quantities may have different t_G's).

Thus we see that the Gaussian theory "explains" the difference between ^{4}He and the superconducting transition. Let us now put down ~~We~~ all the results of the Gaussian fluctuation theory and compare with experiment

Summary of Landau-Ginzburg at the Gaussian level.

Quantity	Exponent	Amplitude ratio
Specific heat C_V	$\alpha = \frac{4-d}{2}$	$\frac{C_+^\alpha}{C_-^\alpha} = n\, 2^{\frac{4-d}{2}}$
M Spontaneous Magnetization	$\beta = 1/2$	—
χ	$\gamma = 1$	$\frac{C_+^\gamma}{C_-^\gamma} = 2$
$M(@t=0) \sim h^{1/\delta}$	$\delta = 3$	—
$\xi = \lvert t\rvert^{-\nu}$ (correlation length)	$\nu = 1/2$	$\frac{C_+^\nu}{C_-^\nu} = \sqrt{2}$

Correlation at critical point

$$\langle \psi_m(r)\, \psi_m(0) \rangle = \frac{1}{\xi_0^2\, r^{d-2}}$$

Away from critical point, power law correlation for $\frac{1}{\Lambda} \lesssim r \ll \xi$; and for $r \gg \xi$; $\langle \psi_m(r) \psi_m(0) \rangle = \frac{e^{-r/\xi}}{r^{\frac{(d-1)}{2}}}$

Critical dimension

n	$d_c^l(n)$	$d_c^u(n)$
1	-	4
≥ 2	2	4

TABLE 3.4

Values of critical-point exponents for selected systems

System	$T < T_c$				$T = T_c$		$T > T_c$		
	α'	β	γ'	ν'	δ	η	α	γ	ν
Fluids									
CO_2	~0·1	0·34	~1·0	—	4·2	—	~0·1	1·35	—
Xe	< 0·2	0·35	~1·2	0·57	4·4	—	—	1·3	—
Magnets									
Ni	α'_s = −0·3	0·42	—	—	4·22	—	0	1·35	—
EuS	α'_s = −0·15	0·33	—	—	—	—	0·05	—	—
$CrBr_3$	—	0·368	—	—	4·3	—	—	1·215	—
Soluble Models									
classical	0 (disc)	$\frac{1}{2}$	1	$\frac{1}{2}$	3	0	0 (disc)	1	$\frac{1}{2}$
Ornstein–Zernike	—	—	—	—	5	0	α_s = −1	2	1
$d = 3$ spherical model	—	$\frac{1}{2}$	—	—	5	0	α_s = −1	2	1
$d = 2$ Ising model	0 (log)	$\frac{1}{8}$	~$\frac{7}{4}$	1	~15	$\frac{1}{4}$	0 (log)	~$\frac{7}{4}$	1
Approximations									
$d = 3$ Ising model (fluid?)	~$\frac{1}{8}$ or ~$\frac{1}{16}$	~$\frac{5}{16}$	~$\frac{5}{4}$ or ~$\frac{21}{16}$	—	~5	~0·041	~$\frac{1}{8}$	~$\frac{5}{4}$	~0·638
$d = 3$ Heisenberg model (magnet?)	—	(~0·345?)	—	—	~5	(~0·03?)	α_s ~ −0·1	~1·4	~0·70

From Stanley, Introduction to Phase Transitions and Critical Phenomena, Oxford

Table 1.2 *Measured values of critical exponents in different systems, belonging to the Ising, XY, and Heisenberg universality classes.*

	Xe	^{4}He	Fe	Ni
Universality class	Ising	XY	Heisenberg	Heisenberg
α	< 0.2	−0.0127 ± 0.0003	−0.03 ± 0.12	0.04 ± 0.12
β	0.35 ± 0.015		0.37 ± 0.01	0.358 ± 0.003
γ	1.3 ± 0.2		1.33 ± 0.015	1.33 ± 0.02
δ	4.3 ± 0.5		4.3 ± 0.1	4.29 ± 0.05
η	0.1 ± 0.1		0.07 ± 0.04	0.041 ± 0.01
ν	≈ 0.57	0.6705 ± 0.0006	0.69 ± 0.02	0.64 ± 0.1

From Herbut, A Modern Approach to Critical Phenomena, Cambridge

Table 1.2. Values of critical exponents

	Xe	Binary fluid	β-brass	^{4}He	Fe	Ni
D	1	1	1	2	3	3
α	< 0.2	0.113±.005	0.05±.06	−0.014±.016	−0.03±.12	0.04±.12
β	0.35±.015	0.322±.002	0.305±.005	0.34±.01	0.37±.01	0.358±.003
γ	$1.3^{+.1}_{-.2}$	1.239±.002	1.25±.02	1.33±.03	1.33±.015	1.33±.02
δ	$4.2^{+.6}_{-.3}$	4.85±.03		3.95±.15	4.3±.1	4.29±.05
η	0.1±.1	0.017±.015	0.08±.07	0.021±.05	0.07±.04	.041±.01
ν	≈ 0.57	0.625±.006	0.65±.02	.672±.001	0.69±.02	0.64±.1

NOTES: From data published in Ahlers (1980), Anders and Stierstadt (1981), Bally *et al.* (1968), Chang *et al.* (1979), Cohen and Carver (1977), Collins (1969), Heller (1967), Hiroyoshi (1980), Kobeissi (1981), Kumar *et al.* (1983), Rocker *et al.* (1971), Rowlinson and Winton (1982), Soeffge (1980), Suter and Hohenemser (1978), Vygovskiy and Yergin (1972). α and β for a binary fluid are for methanol–hexane, while γ, η and ν are for a mixture of trimethylpentane and nitroethane. The data for helium relate to the He I–He II transition. δ for a binary fluid, α for β-brass, and both γ and η for helium have been calculated from the other exponents using formulae from §1.5.1.

From Binney et al., The Theory of Critical Phenomena, Oxford

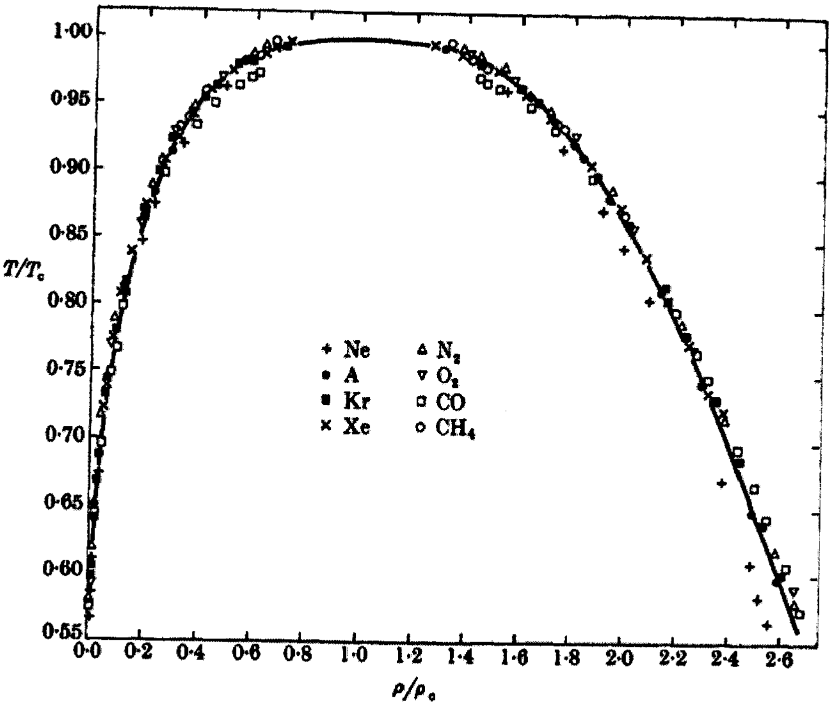


FIG. 1.8. Measurements on eight fluids of the coexistence curve (a reflection of the $P\rho T$ surface in the ρT plane analogous to Fig. 1.3). The solid curve corresponds to a fit to a cubic equation, i.e. to the choice $\beta = \frac{1}{3}$, where $\rho - \rho_c \sim (-\epsilon)^{\beta}$. From Guggenheim (1945).

From Stanley, Introduction to Phase Transitions and Critical Phenomena, Oxford

We now have all the key available data. We see:

① The critical exponents are different from Landau-Ginzburg (Gaussian fluctuation) theory. There is strong dependence of both dimension-d and the number n of components of the order parameter - i.e, there is a dependence on the order parameter symmetry! We have nowhere to hide now! Landau Ginzburg theory (even with inclusion of fluctuations) is not quite right.

② Suppose we now fix d and n. Then we see that there is a remarkable similarity of the critical exponent! This is most vividly seen in the data on liquid gas critical points of various systems. Note that (T_c, P_c) of these various gases are very different and yet in terms of some scaled variables they are all the same. Some how all the microscopic details seems to have been "washed out"

Similarly looking at table 1.2 from Herbut or Binney, one finds that that the critical exponent of Fe and Ni are quite close to each other. Note and $T_c^{Fe} = 1043K$ while $T_c^{Ni} = 627$ K, this is very different; the electrons have different effective masses etc! But none of this matters for the critical exponents

The phrase that is used to describe such a thing is called "Universal behavior". Systems when reived suitably ~~loose~~ near a critical point lose all sense of their "individuality" (Fe or Ni) and show universal behavior. There are various types of universal behaviors and ~~those~~ this is determined by the spatial dimension and the order parameter symmetry (O(n) in our case). As soon as these two things are specified, the critical behavior ~~#~~ near a second order phase transition point is completely defined. Such different class of universal behaviors are called universality classes. For example the liquid-gas critical behavior of all the systems shown in Fig 182 two pages back belong to the Ising universality class in $d=3$. Similarly the ferro magnetic critical point of Fe and Ni belong to the Heisenberg (O(3)) universality class in $d=3$.

③ There is also some new ~~ol~~ object called η in the data that we have seen. One can actually measure the correlation of fluctuations at the

Critical pt: $\langle \psi(r)\, \psi(0) \rangle \overset{(GLG)}{\underset{t=0}{=}} \frac{1}{\xi_0^2\, r^{d-2}}$.

But what is <u>measured</u> is not this! ~~We~~ A power law is seen! But with a different exponent

$$\langle \psi(r)\, \psi(0) \rangle \overset{Expt}{\underset{t=0}{=}} \frac{1}{r^{d-2+\eta}} \; !$$

This means the result of GLG which comes essentially from dimensional considerations is <u>strongly</u> modified. In other words

$[\phi(r)] \sim L^{-d/2}$ does not seem to hold!!!

We say the $[\phi(r)]$ picks up an anomalous dimension; and η itself is sometimes called the anomalous dimension.

This is a really puzzling thing!!
Where ~~thus~~ does the anomalous dimension come from?

~~Other~~ Key questions to think of are

① What is the origin of universal behavior? Where does the "microscopics go"?

② Why are the critical exponents different from LGG theory?

③ What is the origin of anomalous dimensions?

This will be the subject of the next chapter.