Chapter 4:

Scaling and Renormalisation Group

We have seen thus far that near a critical point there are some characteristic divergences that are described by the critical exponents.

$$C_V = C_\pm^{\alpha} |t|^{-\alpha}$$

$$M = |t|^{\beta}$$

$$\chi = C_\pm^{\chi} |t|^{-\gamma}$$

$$M(t=0) = h^{1/\delta}$$

$$\xi = C_\pm^{\nu} |t|^{-\nu} \quad \text{(Correlation length)}$$

which defines five (5) critical exponents

We have also seen that experiments suggest that the correlation function

$$G^{d}(r) \sim \frac{1}{r^{d-2+\eta}}$$

has a power law behavior at the

critical point. If we look at this in fourier space

$$G^d(k) \sim \frac{1}{k^{2-\eta}}$$

[Indeed $G^d(r) = \int d^dk \, \frac{e^{i\vec{k}\cdot\vec{r}}}{k^{2-\eta}} \sim \frac{1}{r^d r^{-2+\eta}}$

$$\sim \frac{1}{r^{d-2+\eta}}].$$

What should we expect away from the critical point clearly

$$\underset{t\neq 0}{G^d(k)} \sim \frac{1}{k^{2-\eta} + \xi^{-(2-\eta)}}$$

and as $\xi \to \infty$ when $t \to 0$ the ~~we~~ remt lends to $\frac{1}{k^{2-\eta}}$.

~~The remark~~. Including η, we see that there are six critical exponent $\alpha, \beta, \gamma, \delta, \nu, \eta$. The remarkable aspect is that there exponent depend only on the spatial dimension and the symmetry group and not on the microscopic details of the Hamiltonian

This is called universality, and associated with a different universality classes, there are distinct set of critical exponent. And the remarkable thing here is the anomalous dimension η. → this is quite puzzling since it challenges some of our most basic ideas such as dimensional analysis.

How do we understand this and what ~~sh~~ should we do to proceed? Fortunately our seniors had thought about this question: They looked at experiment – requested nature to part with ~~what we~~ ~~the~~ her secrets! They then made a startling hypothesis and worked things out based on this hypothesis..

To see what the hypothesis is, we need some mathematical definitions

Generalized Homogeneous Function:
Consider a function of many variables $F(x, y, \ldots)$. Now a function is a generalized homogeneous function with ~~indices~~ degrees $(\mu_x, \mu_y, \ldots)$ if

$$F(\cancel{x}\, a^{\mu_x} x,\ a^{\mu_y} y \ldots) = a F(x, y \ldots y)$$

For example

$$F(x,y) = x^{\pi} y^{e}$$

is a homogeneous function with $\mu_x = \frac{1}{2\pi}$ and $\mu_y = \frac{1}{2e}$. Back to physics!

<u>Scaling Hypothesis</u>: Widom, Fisher and other physicists, looked at lots of experimental data and made the following hypothesis. call the scaling hypothesis or the homogeneity assumption.

The state ment is:

The singular part of free energy near a ~~pt~~ 2nd order critical point is a generalized homogeneous function of its variables. The free energy $f(t,h)$ (note ~~[illegible]~~ that this is the actual singular part of the free energy, not the landau type one) satisfies

$$\boxed{f(a^{\mu_t} t, a^{\mu_h} h) = a f(t,h).}$$

This sounds rather bizarre and far reaching. and was arrived at by long and pain staking work of understanding experiments. Rather than dwell on the motivations of this, let us see what this hypothesis implies. In other words we make a falsifiable ~~I have wanted you~~ prediction. Upshot is that, if this hypothesis is true, then the critical exponents are not independent.

Note that we can chose a to be anything, so we could choose $a = t^{-1/\mu_t}$

$$\Rightarrow t^{+\frac{1}{\mu_t}} f(1, t^{-\mu_h/\mu_t} h) = f(t, h).$$

Thus

$$c_V = \left.\frac{\partial^2 f}{\partial t^2}\right|_{h=0} \simeq t^{\frac{1}{\mu_t}-2}$$

$$\Rightarrow \quad \alpha = 2 - \frac{1}{\mu_t}$$

$$m = -\left.\frac{\partial f}{\partial h}\right|_{h=0}$$

$$\sim t^{\frac{1-\mu_h}{\mu_t}}$$

$$\Rightarrow \quad \beta = \frac{1-\mu_h}{\mu_t}$$

$$\chi = -\left.\frac{\partial^2 f}{\partial h^2}\right|_{h=0} \Rightarrow \frac{2\mu_h - 1}{\mu_t} \equiv \gamma$$

To obtain δ, we use a slightly different trick

$$f(t,h) = h^{+1/\mu_h} f\left(h^{-\frac{\mu_t}{\mu_h}} t, 1\right)$$

Now $\quad m = \frac{\partial f}{\partial h} \Rightarrow + \frac{1}{\mu_h} - 1 = \frac{1}{\delta}.$

We thus see that α, β, γ and δ are completly determined by μ_t and μ_h. This means that there ~~For exam~~ are constraints equations that can be obtained by eliminating μ_t, μ_h

For example

$$\alpha + 2\beta + \gamma =$$

$$2 - \frac{1}{\mu_t} + \frac{2(1-\mu_h)}{\mu_t} + \frac{2\mu_h - 1}{\mu_t} = 2$$

$$\alpha + 2\beta + \gamma = 2 \quad \text{[Rushbrooke's Law]}.$$

Consider now

$$\delta - 1 = \frac{2\mu_h - 1}{\mu_h - 1} = -\frac{\beta}{\gamma}$$

$\Rightarrow$

This implies

$$\gamma(\delta - 1) + \beta = 0$$ Widom's Law.

Now we know that

$$\langle \psi_m(\vec{r})\, \psi_m(0) \rangle = G_m(\vec{r})$$

(let us drop the m-label) stands for a susceptibility for a δ-function magnetic field applied at the origin. We thus obtain the susceptibility for a homogeneous magnetic field to be

$$\chi = \int d^d r \; G_m(\vec{r}) = G(\vec{k}=0)!$$

$$= \left. \frac{1}{k^{2-\eta} + \xi^{-(2-\eta)}} \right|_{\vec{k}=0}$$

We get $\Rightarrow$

$$\chi = \xi^{2-\eta}$$

$$= |t|^{-\nu(2-\eta)}$$

But

$$\chi = |t|^{-\gamma}$$

$$\Rightarrow \boxed{\gamma = \nu(2-\eta)}$$ Fisher Law!

Note that these scaling relations (also called as "laws") do not depend explicitly on the spatial dimensions.

Fisher, inspired by Josephson, argued that ~~~~ the singular part of the free energy density scales with ~~~~ the correlation length ξ. The idea is that near the critical point ξ is the only length scale, so that

$$f(t, \cancel{h=0}) \sim \frac{1}{\xi^d}$$

We get ~~~~ $f(t, h=0) \sim |t|^{+\nu d}$

Thus $\underline{-\nu d + 2} = \alpha$ or $\boxed{-\nu d = \alpha - 2}$

$\boxed{\nu d = 2-\alpha}$

This is the Josephson law, and depends ~~~~ explicitly on the spatial dimension d. ~~Note that~~ Such scaling laws are called ~~~~ as hyperscaling laws and ~~~~ do not hold generically. For example in $d > 4$, this is not true. Roughly speaking, if the free energy is dominated by fluctuation then ~~~~ this law will hold.

Since the free energy has to go to zero at the transition the idea then is that it goes to zero as $\frac{1}{\xi^d}$. Let us say that Josephson holds.

Now we have found ~~four~~ scaling laws relating six critical exponent. We can check easily with either experiment or ~~ea~~ the analytical solution of the Ising model to see if these relations hold. Indeed one finds that ~~on~~ the exponents calculated from the exact ~~exponent~~ Ising theory indeed satisfy these relations. This suggests that the idea of the scaling hypothesis is indeed plausible.

If one thinks about this, one finds that the situation is rather unsatisfactory. Scaling hypothesis produced some relationship between the critical exponent. Okay good! But it do not tell ~~to~~ us how to calculate the exponents, nor does tell us *why* there is universality or the anomalous dimension. Indeed the question is what does the scaling hypothesis *teach* us?

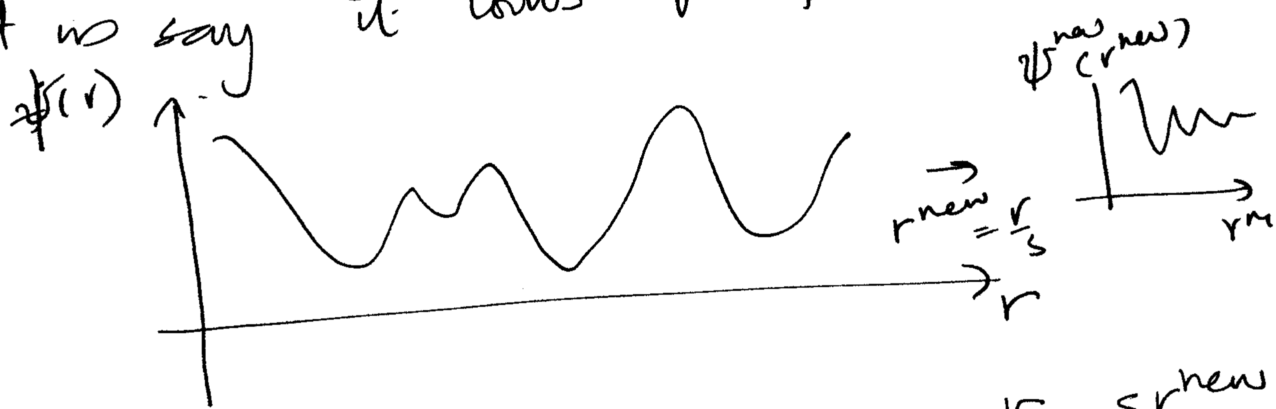
Our seniors in the 1960s had a ball of a time thinking about these things. Due to the efforts of many, the ideas become clearer over a period of time. Motivated by the works of Kadanoff, Migdal, Fisher et. al, and taking inspiration from numerical calculation and things, ~~[illegible]~~ Wilson (who just recently passed away) ~~stated~~ not only stated precisely a single principle that brings out the physics of the scaling hypothesis, but also ~~allows~~ ~~for the~~ provides an explanation of universality, and even allows the calculation of the critical exponents including the ~~scale~~ anomalous dimension. ~~So what [illegible] say?~~ So what is this ~~law~~ principle or law?

At a critical point a thermodynamic system obtains a <u>new symmetry</u>, called scale invariance or dialatational symmetry.

What do we mean by scale invariance or dialatational symmetry? Our system has a Hamiltonian $\mathcal{H}$. Suppose we now change the scale of measurement of position by a factor s. Then

$$\vec{r}_{new} = \frac{\vec{r}}{s}$$

In otherwords what was originally 1 meter is now $1/s$ meter in this new scale. Note that this changes the arena ~~is~~ and consequently the ~~Hamiltonian~~ H changes to H'. Scale invariance means $H = H'$! In other words the scaling transformation is a symmetry of the Hamiltonian if $H = H'$ is true for all s. But what does this really mean? This means that the fluctuations at the critical point will be scale invariant. When do we say that the fluctuation is scale invariant. Take a "snapshot" of the system, and look at the fluctuations, $\psi(r)$. Let us say it looks ~~like~~ ~~f(r)~~ correlation function $f(r)$



Now we change $r \to r^{new}$ via $r = s r^{new}$.
correlation
In this scale the, fluction looks like
$f^{new}(r^{new}) = f(s r^{new})$, Now
scale invariance ~~will~~ means that $f(s r^{new})$
$f^{new}(r^{new})$
$= s^{\alpha} f(r^{new})$. In other words
$= s^{\alpha} f(r)$! Note that roughly what we are saying is that the ~~correlations are~~ fluctuations are "fractal like", and since they do not have a scale, the correlation ~~are~~ is a homogeneous function.

Note that $f(r) = r^{-\alpha}$ (a <u>power law</u>) produces scale invariant correlation functions! This is certainly consistent with experiment.

Now, how do we use this "law" of scale invariance at the critical point to see the ~~scale invariance~~ the scaling laws emerge as well as universality?

To do this let us be a bit more concrete. Consider the O(n)) model on a d-dimensional continuum.

The Hamiltonian of the system generically contains all terms allowed by O(n) symmetry. For example

$$\beta\mathcal{H} = \int d^d\vec{r} \; a_{20}(\vec{\Phi}\cdot\vec{\Phi}) + a_{40}(\vec{\Phi}\cdot\vec{\Phi})^2 + a_{21}(\vec{\nabla}\vec{\Phi})^2 + a_{22}((\vec{\Phi})\cdot(\vec{\nabla}^2\vec{\Phi}))$$

$$+ \ldots \text{ all terms allowed by symmetry.}$$

Such a Hamiltonian is described by a set of (infinite) numbers like $(a_{20}, a_{40} \ldots)$. We can denote this by a point P in the "Hamiltonian space",

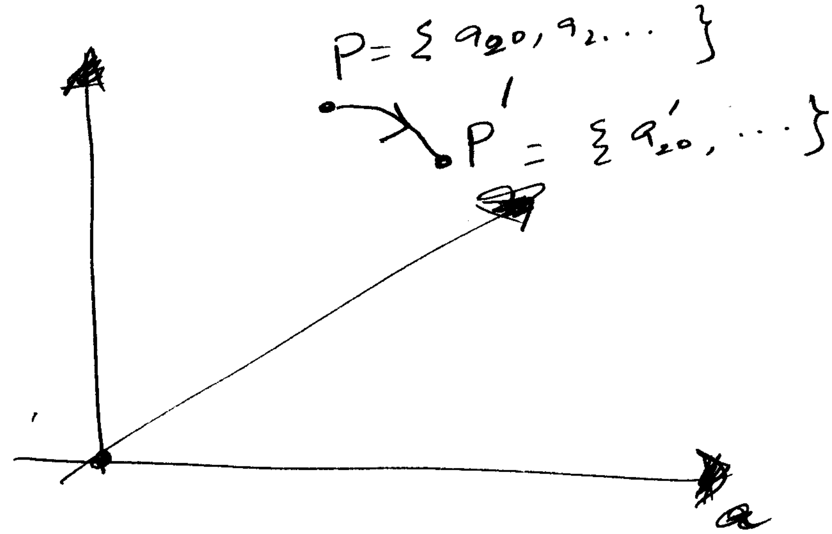
We now write $\beta\mathcal{H}(P)$.

Now what happens when we do the scale transformation $\vec{r}^{\,new} = \dfrac{\vec{r}}{s}$ $(s>1)$

The Hamiltonian changes, but its form is unchanged as it is determined solely by symmetry

$\cancel{\beta\mathcal{H}}$ $\beta\mathcal{H} \to \beta H'$

In other words the coefficients change from $P = \{a_{20}, a_{21}, \dots\}$ to $P' = \{a'_{20}, a'_{21}, \dots\}$. One can view the scale transformation as inducing a "flow" in the Hamiltonian space



This flow ~~can be~~ is a short form for

$$\beta\mathcal{H}(P) \longrightarrow \beta\mathcal{H}(P').$$

We denote this transformation as $R_s(P) = P'$. The transformation R_s satisfies

$$R_{s_1 s_2}(P) = R_{s_1}(R_{s_2}(P)) = R_{s_2 s_1}(P).$$

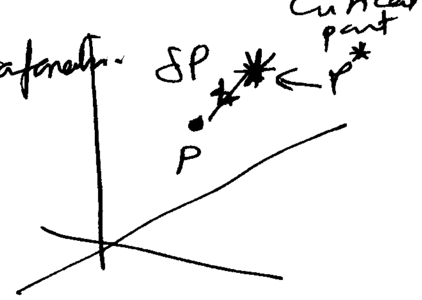
$$R_{s=1}(P) = P, \qquad R_{s_1 s_2} = R_{s_1} R_{s_2}$$

This transformation has a semi-group structure, and is called

the Renormalization group. (RG) transformation.
Now, using the ~~\$~~ statement that the critical Hamiltonian is scale invariant, ~~Let~~ we obtain some relation. Let P^* be a fixed point of R_s, i.e.,

$$R_s(P^*) = P^* \; (\forall s>1)$$ thus P^* describes a critical Hamiltonian.

A critical point is a fixed point of an RG transformation.



Let us consider a point P in the neighbourhood of p^*. What is $R_s(P)$?

taking $P = P^* + \delta P$

$$R_s(P) = R_s(P^* + \delta P)$$

so that $$R_s(P) = P^* + \underbrace{L_s(P^*)}\,\delta P$$

matrix

L_s (first derivative of R_s at P^*)

Thus.
$$R_s(P) - p^* = L_s(P^*)\,\delta P$$

Now that $L_s(P^*)$ is a property of the fixed point p^*. Let us find the right eigenvectors of this matrix, ~~calling~~

$$L_s(P^*)\,O_i = \lambda_i(s)\,O_i$$

O_i are some direction in the Hamiltonian space and ~~___~~ λ_i are the corresponding

O_i are the corresponding eigenvectors that depend on s, ie

$$L_s(P^*) = \sum_i{}' \lambda_i(s)\ O_i \otimes O_i$$

Now

$$L_s(P^*)\,\delta P = L_s(P^*) \underbrace{\sum_i{}' g_i O_i}_{\text{expanding in the } O_i\text{'s}}$$

$$= \sum_i{}' g_i\ \lambda_i(s)\, O_i$$

Note $L_{s_1} L_{s_2} = L_{s_1 s_2}$

$$\Rightarrow \quad \lambda_i(s_1 s_2) = \lambda_i(s_1)\,\lambda_i(s_2).$$

clearly $\lambda_i(s)$ of the form s^{μ_i}

μ_i are some set of numbers.

$$L_s(P^*)\,\delta P = \sum_i{}' g_i\ s^{\mu_i} O_i$$

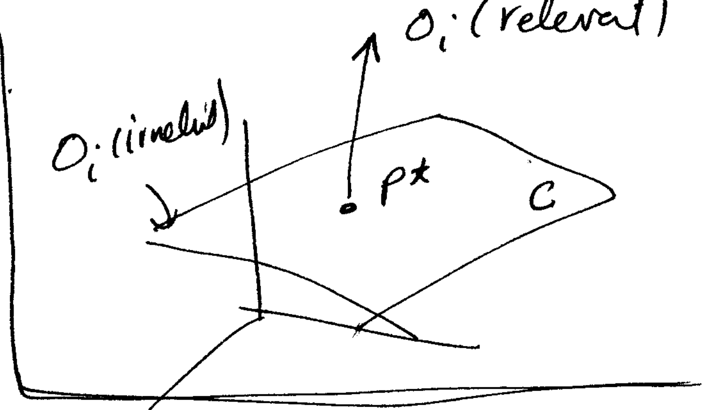
Note that $s > 1$. So if

$$\mu_i > 0 \quad \text{(relevant)}$$

O_i will grow, and take us further away from the critical point. On the other hand $\mu_i < 0$ (irrelevant)

O_i's influence falls, which if $\mu_i = 0$ (marginal),

Now let C be the hypersurface near p^* that spans over the irrelevant direction. If $\delta P \in C$, the $R_s(P)$ will bring us closer to the critical point. On the other hand, if $P \notin C$, and is along a relevant direction ~~p^*~~ R_s will take us away from the critical point.



Quite interestingly universality emerges quite nicely. It turns out that most directions are irrelevant, and realistic systems, there are only a few relevant directions (such as temperature and magnetic field). It is ~~th~~ these few relevant directions that determine the critical properties, including the anomalous dimension. While the ~~are~~ statement that most ~~direction~~ O_i are irrelevant is rather sweeping, if we accept this we understand universality.

We can push this even further. Note that under R_s $\beta H \to \beta H'$ and $\phi \to \phi'$ (the fields). Now

$$Z = \int \mathcal{D}[\phi] e^{-\beta H} = \int \mathcal{D}[\phi'] e^{-\beta H'}$$

Due to this the free energies must obey

$$V f(P) = V' f^{*}(P').$$

Since $V' = \frac{V}{s^d}$, we obtain

$$f(P) = s^{-d} f(P').$$

~~Thus~~ If P is near P^*

$$f(\delta P) = s^{-d} f\left[L_s(P^*)\delta P\right]$$

or

$$f[\{g_i\}] = s^{-d} f\left[\{s^{\mu_i} g_i\}\right]$$

Now we see that we need to keep only relevant operators (marginal ~~even~~ also ~~does~~), so that the irrelevant ~~parts~~ operators do not have any say. We herefore obtain

$$f(\{g_i\}) = s^{-d}(\{s^{\mu_i} g_i\})$$

Relevant

But, hey, we ~~know~~ know this! This is the Scaling Hypothesis!! Thus the idea that the critical point ~~not only~~ is scale invariant, not only proves an idea of universality but also produces the scaling hypothesis as a corollary! This is indeed a quite successful idea in this regard, ~~and and~~ as it automatically explains the scaling laws.

Note that the picture also ~~says~~ gives ~~the~~ a nice relationship for the correlation length ξ.

$$\xi(\{g_i\}) = s\,\xi(\{s^{\mu_i} g_i\})$$

Relevant.

Thus everything is crucially determined by μ_i (Relevant). How do we determine these? We need to construct R_s explicitly. ~~Th~~ Here ~~is~~ is where Wilson made his crucial contribution. ~~Not~~ He formulated a method that not only provided μ_i for a generic model but also solidified the ideas. But before we see Wilson's formulation, we will visit Kadanoff-Migdal.

Before we do an actual calculation, it is good to familiarize ourselves with some standard terminology.

Note that

$$P(s) = P' = R_s(P)$$

is a well defined RG transformation for all $s \geq 1$. Even so, it is very difficult to construct this for any $s > 1$! Inspired by Wilson's work, it became clear that reliable construction of R_s can be obtained for $s \gtrsim 1$. The trick is to write $s = e^l$ where $l \approx 0$ and this $\frac{ds}{s} = dl$. One then writes the RG equation as a flow equation ie

$$\underbrace{s \frac{d}{ds} R_s(P)}_{\beta(P)} = s \frac{dP^*}{ds} = \frac{dP}{dl}$$

or

$$\frac{dP}{dl} = \beta(P).$$

$\hookrightarrow$ The beta function

Note that β is a column matrix

with as many entries as there are numbers in P. Wilson's wonderful ideas provides ways of calculating $\beta(P)$ directly (without explicitly worrying about R_s for any s). Note that the critical or fixed points of the RG flow satisfy

$$\beta(P^*) = 0.$$

In our calculations we shall use these ideas extensively

Now we turn to the key question: How do we construct R_s for a generic system? ~~This~~ Historically, this turned out to be a quite difficult problem. The generic framework for this was developed by Wilson, which we shall see a bit later. Generically the RG methods can be put together into three classes.

① Real space methods (Pioneered by Kadanoff et al.)

(2) Field theoretic / k-space methods (primarily developed by Wilson)

(3) Numerical methods that use RG ideas. (We will not touch this subject).

Let us begin by looking at a real space method. The simplest model with an exact solution is the 1d-Ising chain and it is not a bad idea to start with that.

<u>RG transformation of the 1D Ising chain:-</u>

Before we do this let us briefly recall the results of the exact solution.

We need to be a bit careful here since $T_c = 0$; so $\frac{t - T_c}{T_c}$ will not quite work. We can use $t = e^{-2J/T}$ instead. In terms of this we see that

$$f^s = -T \ln \left(2\cosh\left(\frac{J}{T}\right)\right)$$

$$= -T \ln\left(e^{J/T}\left(1+e^{-\frac{2J}{T}}\right)\right)$$

and the correlation length

$$\xi \sim \left[\ln \frac{1}{\tanh J/T}\right]^{-1} = t^{-1}$$

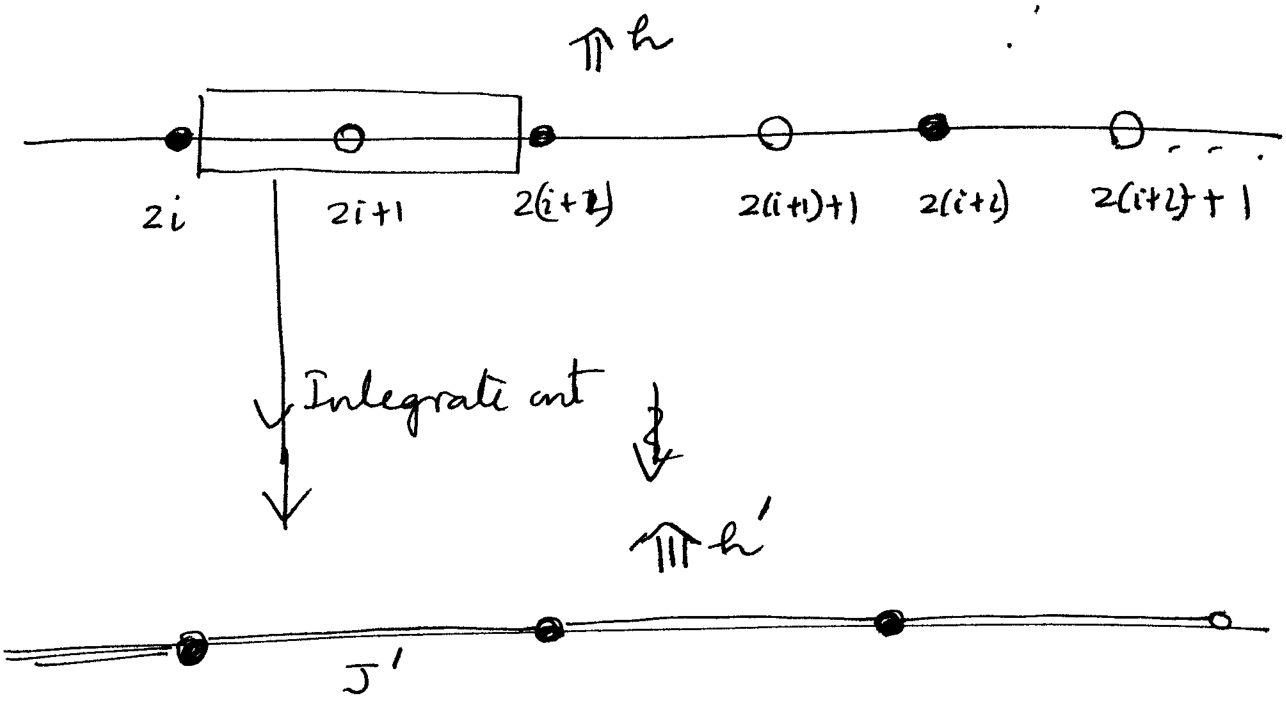
$\Rightarrow \nu = 1.$

Let us see how to do the RG. here.
The process is called decimation (
decimation is a term in old english, where
a conquering army killed one in ten of
the captured soldiers at random!)
Take the Ising chain (Infinite chain

$$\beta H = -J\sum_i{}' S_{i+1}S_i - h\sum_i{}' S_i + \sum_i{}' C$$

~~what is~~ C. Note J and h are dimensionless
and C is a constant energy allowed by
symmetry.
Now what we do is we rewrite this
Hamiltonian ~~as~~ by keeping the
~~dod~~ odd and even sites.

$$\beta H = \sum_i{}' \overbrace{-\left(J S_{2i+1}S_{2i} + \frac{h}{2}(S_{2i+1}+S_{2i}) + C\right)}^{H_{2i}} \underbrace{-\left(J S_{2(i+2)} S_{2i+1} + \frac{h}{2}(S_{2i+2}+S_{2i+1}) - C\right)}_{H_{2i+1}}.$$



$j = i$

Now we "decimate" the odd sites and expect to obtain a Hamiltonian of the form

$$(\beta H)' = -\sum_j{}' J'(s_{j+1} s_j) - h'\sum_j{}' s_j - \sum_j{}' c'$$

Since we are decimating one in every 2 sites, this is equivalent to choosing $s = 2$. In an earlier notation

$$R_{s=2}[J, h, c] = R_2[\{J, h, c\}] = \{J', h', c'\}.$$

But is this even possible or are we simply being too optimistic?

Let us explore.

For the original system

$$Z = \sum_{S_{even}}^{1} \sum_{S_{odd}}^{1} e^{-\sum_i^1 (H_{2i} + H_{2i+1})}$$

$$= \sum_{S_{even}}^{1} \sum_{S_{odd}}^{1} \prod_i e^{-(H_{2i} + H_{2i+1})}$$

$$= \sum_{S_{even}}^{1} \prod_i \underbrace{\sum_{S_{2i+1}}^{1} e^{-(H_{2i} + H_{2i+1})}}$$

$$\underbrace{= e^{J(S_{2i} + S_{2i+2}) + \frac{h}{2}(S_{2i} + S_{2i+2}) + h + 2C} + e^{-J(S_{2i} + S_{2i+2}) + \frac{h}{2}(S_{2i} + S_{2i+2}) - h + 2C}}_{= M(j)}$$

$$= \sum_{\{S_j\}}^{1} \prod_j M(j)$$

where ~~i~~ $j = i$

$S_{2i} \to S_j$

$S_{2i+2} \to S_{j+1}$

We will now show that $M(j)$ has same matrix elements as $e^{-H_j'}$

$$e^{-H_j'} = e^{J' S_j S_{j+1} + \frac{h'}{2}(S_j + S_{j+1}) + c'}$$

For this to work we need to show that there are unique J', h', c' such that the matrix ~~to~~ element of $M(j)$ are identical to $e^{-H_j'}$

To see this write out the matrix element explicitly

S_j, S_{j+1}	$M(j)$	$e^{-\beta H'_j}$
$\uparrow\ \uparrow$	$e^{2J+2h+2c} + e^{-2J+2c}$	$e^{J'+h'+c'}$
$\uparrow\ \downarrow$	$e^{h+2c} + e^{-h+2c}$	$e^{-J'+c'}$
$\downarrow\ \uparrow$	$e^{h+2c} + e^{-h+2c}$	$e^{-J'+c'}$
$\downarrow\ \downarrow$	$e^{-2J+2c} + e^{2J-2h+2c}$	$e^{J'-h'+c'}$

To solve these equations define

$$e^{-2J} = t \qquad e^{-\cancel{2}h} = u, \quad e^{2c} = v$$

We get

$$t^{-1}u^{-1}v + t\,v = \sqrt{(t')^{-1}(u')^{-1}v'}$$

$$(u^{1/2}+u^{-1/2})\,v = (t')^{-1/2}\,v^{1/2}$$

$$(t u + t^{-1}u^{+1}v) = \sqrt{(t'^{-1})(\cancel{t}u')^{+1}v'}$$

we get:

$$(t')^{-\cancel{4}2} = \frac{(t^{-\cancel{4}}u^{-1}+t)(t+t^{-1}u^{+1})}{(\sqrt{u}+\sqrt{u^{-1}})^2}$$

$$(u')^{-\cancel{1}} = \frac{(t^{-1}u^{-1}+t)}{(t+t^{-1}u^{+1})}$$

$$(v')^2 = v^4\,(t^{-1}u^{-1}+t)(t+t^{-1}u^{+1})(\sqrt{u}+\sqrt{u^{-1}})^2$$

$$R_{s=2}[\{t,u,v\}] = \{t',u',v'\}.$$

We thus see a beautiful result!

The decimated hamiltonian does correspond to an Ising model with nearest neighbour interaction! ~~But with~~. The key is that the coupling constants have changed!

We also see some other neat thing:

① t', u' depend only on t and u. ~~and~~

This is because v depends only on the constant part of the Hamiltonian, and does not affect the coupling as we do the scale transformation.

Suppose we start the RG with $h=0$. This means that $u = 1$ to start. But ~~thus~~ if $u=1$ at an ~~iteration~~ iteration of the RG, u' at the next iteration will also be 1 and will remain 1 always! This means that an Ising model (nearest neighbor) will remain one nearest neighbor Ising upon scaling and will not produce a symmetry breaking field.

Let us set $h=0$. Now the RG recursion for t is

$$(t')^{-1} = \frac{(t^{-1}+t)}{2} ;$$

$$\frac{1}{t} = \frac{1}{2}\left(\frac{1}{t}+t\right)$$

$$t=1 \text{ or } t=\infty$$

We want the fixed point. ~~solve~~

~~$\frac{1}{t} = \frac{t + t^{-1}}{2}$~~

Now recall $t = e^{-2J}$

$$e^{2J'} = \frac{e^{2J}+e^{-2J}}{2}.$$

or $J' = \frac{1}{2}\ln\cosh 2J$

The fixed points are obtained by solving

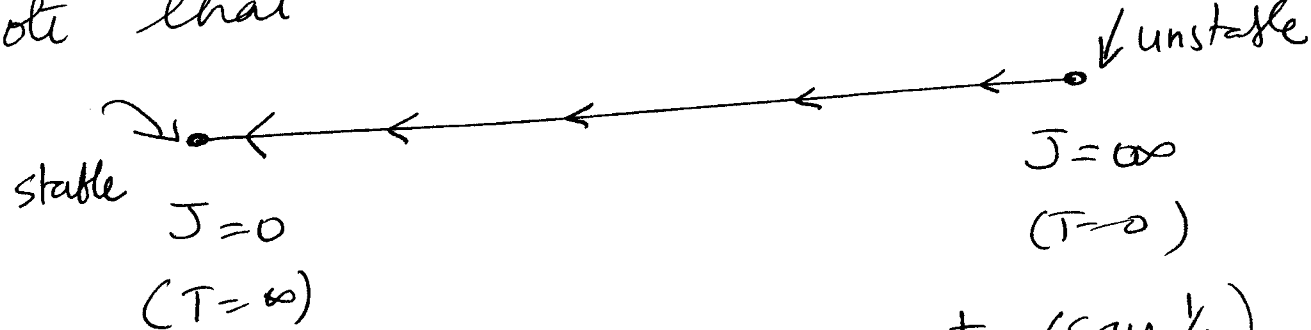
$$J = \frac{1}{2}\ln\cosh 2J .$$

There are two solution $J=0$ and $J=\infty$!

$J=0$ corresponds to ∞ temperature at which the spins decouple. This state has a zero correlation length. (This state corresponds to $t=1$)

~~One~~ The $J=\infty$ solution corresponds to the critical point at $t=0$, and is the $T=0$ and $h=0$.

Note that



Suppose we start with a finite t (say $\frac{1}{2}$)

the $(t')^{-1} = \frac{1}{2}\left(\frac{1}{2}+2\right) = \frac{5}{4} \Rightarrow t' = \frac{4}{5}$

which takes it closer to $t=1$. This is true for any $t>0$. Thus the fixed point $J=0$ is stable to temperature, while $J=\infty$ is unstable

What happens when $h \neq 0$?
Let us look at two limits.

① $J = 0$ or $t = 1$.

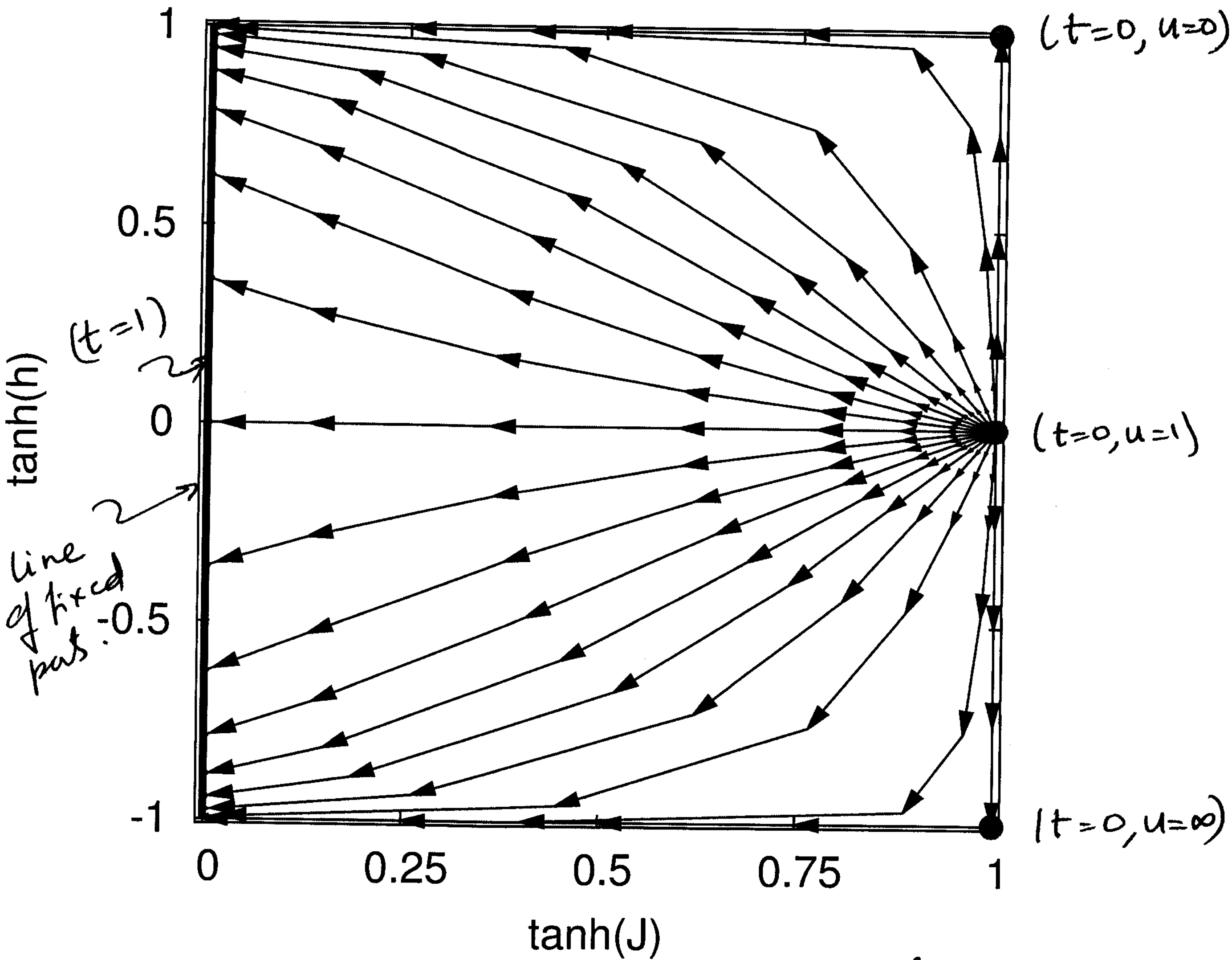
Here $(t')^{-2} = \dfrac{(\frac{1}{u}+1)(1+u)}{u(1+\frac{1}{u})^2} = 1$!

$(u')^{-1} = \dfrac{(1+u^{-1})}{(1+u)} = \dfrac{1}{u}$!

Note that neither t nor u flows! So we have a fixed line or a line of fixed points which correspond to ∞ Temperature and magnetic fields. These fixed points have zero correlation length.

② $J = \infty$ or $t = 0$

here $(t')^{-2} = \infty \Rightarrow t' = 0$



RG. flow digram of the 1D Ising chain

---

and $(u')^{-1} = \frac{1}{u^2}$

which gives two additional fixed points $(t=0, u=0)$ and $(t=0, u=\infty)$. These correspond to the ~~ff~~ fully magnetized states at $T=0$ with ↑ or ↓ polarization. All these fixed points are unstable.

What happens for a generic t and u?
We see from the flow diagram that the system flows to a point on the line of fixed points. Physically this corresponds to the pure paramagnet in a large magnetic field. - this is in the paramagnetic phase. ~~In fact~~
There are two other fixed points with $(t=0,\ u=0)$ and $(t=0,\ u=\infty)$. These correspond to the fully magnetized ground states. Even these fixed points are unstable. The most interesting fixed point for us is the $(t=0,\ u=1)$ fixed point.
We can use the RG flow equations to obtain the critical exponents.
$R_{s=2}(t,u)$ has to be linearized near
$p^*(t=0,\ u=1)$ we find $\delta P = (t, \underbrace{u-1}_{\delta})$

$$R_{s=2}(t, \delta u) \cong (2t,\ 2\,\delta\mu) \text{ (Using Mathematica)}$$

$$R_s^2\,(t, \delta u) = (4t,\ 4\,\delta\mu)$$

$$R^2_{s=2} = R_{s=4}(t,\delta u) = (4t, 4\delta u)$$

We see from this that

$$\mu_t = 1 \quad \text{and} \quad \mu_h = 1.$$

We can now calculate the critical exponent from our scaling hypothesis which is

$$f(s^{\mu_t} t, s^{\mu_h}\delta u) = s^{d=1} f(t,\delta u)$$

$$\Rightarrow \qquad \alpha = 2 - \frac{1}{\mu_t} = 1$$

$$\beta = \frac{1-\mu_h}{\mu_t} = 0$$

$$\gamma = \frac{2\mu_h - 1}{\mu_t} = 1$$

$$\delta^{-1} = \frac{1}{\mu_h} - 1 = 0$$

From hyper scaling

$$\nu = \cancel{d}\,\frac{2-\alpha}{d} = 1$$

From Fisher $\quad \eta = 1$

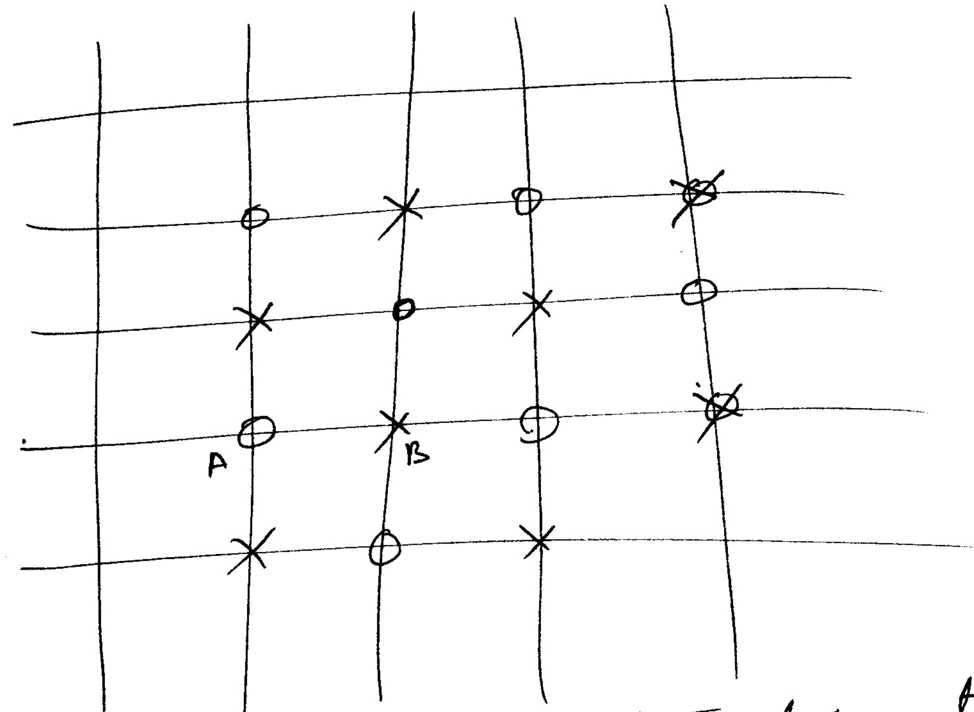
We have thus obtained all the critical exponent (including η). (note the $\eta = 1$ makes our formulas: ~~our formula to~~ Sense.

$$G(r) = \frac{e^{-r/\xi}}{r^{d-2+\eta}} \quad \underset{=}{\overset{d=1,\,\eta=1}{}} \quad e^{-r/\xi}.$$

We see that RG produces the exact critical exponent. This ~~is~~ is a very lucky ~~ex~~ situation since it is a rare example of a case where we can pull off the RG equations exactly.

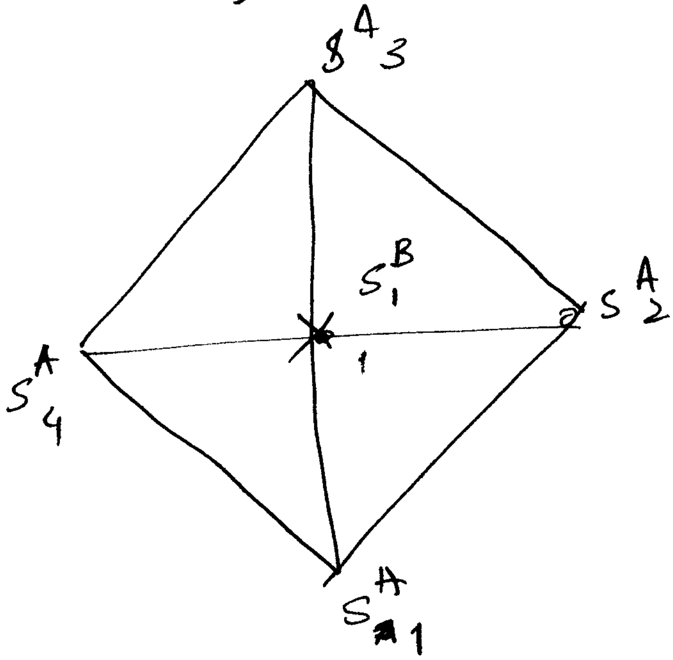
But this is a ~~rather~~ very instructive example, if a bit boring. We will now move on to higher dimensions.

Let us try to implement the decimation idea on a square lattice.

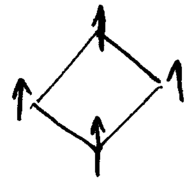


The idea would be to integrate out all the spins on the B sublattice. This is the natural generalization of ~~th~~ what happened in the 1-D model.

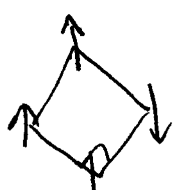
The difficulty can be illustrated by considering a simple A plaquette



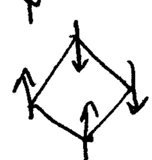
Suppose we integrate out s^B_1. We will generate several types of terms. We can see that ~~the~~ the effective Hamiltonian will be determined by 4 independent parameters. To see this note that s^A_i has $2^4 = 16$ different spin configuration. They are of the following type



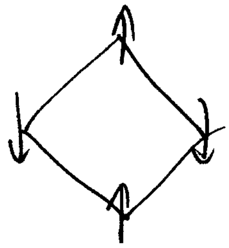
→ all ~~bar~~ up. (2 states) Z_2 related



→ 1 - flip [8 states]



→ 2 spin flip [4 states)



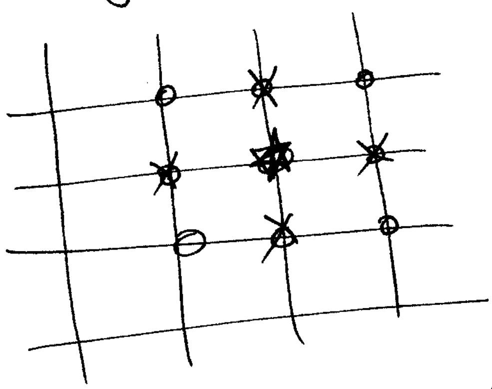
$\rightarrow$ 2 spin flips (2 states).

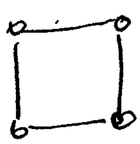
Note that matching $e^{-\beta H'}$ with $\mathrm{Tr}_{S_i^B}\, e^{-\beta H}$ will then have four independent equations. Thus $\beta H'$ must have four independent terms. Thus the new Hamiltonian will have many terms;

$$\beta H' = \underset{\text{constant}}{\underset{\uparrow}{C'}} - J_1' (S_1^A S_2^A + \cdots) - \underset{\text{2nd neighbor}}{\underset{\uparrow}{J_2'}} (S_1^A S_3^A + \cdots) - K (S_1^A S_2^A S_3^A S_4^A).$$

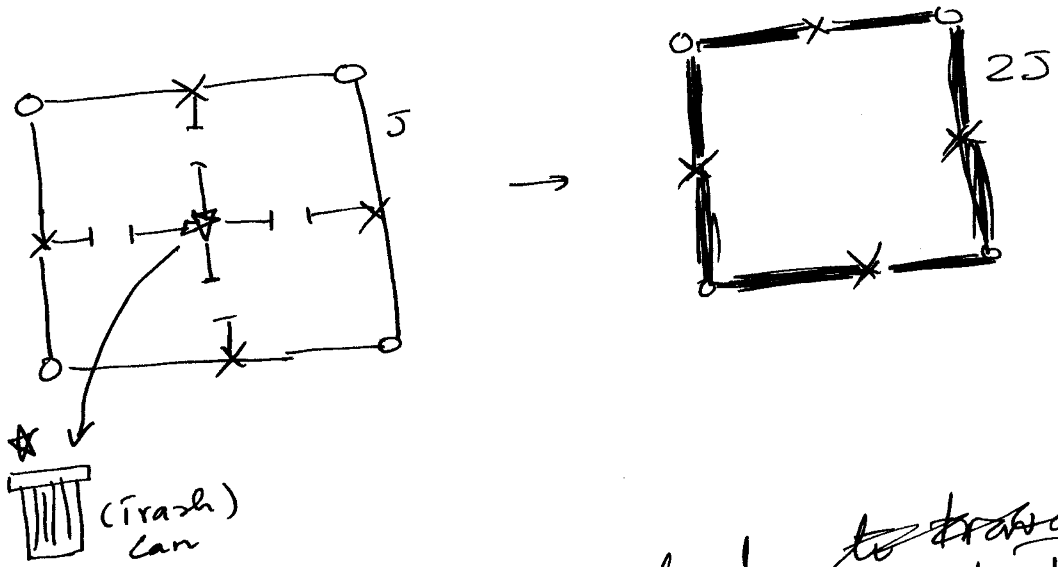
We thus see that we get a very complicated Hamiltonian! We see immediately that the simple decimation idea will not work in 2D and higher D. How do we proceed? ~~[illegible]~~ Clearly one has to give up the dream of finding an exact RG transformation

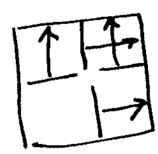
A simple idea to break the logjam was provided by Migdal and developed by Kadanoff. The key hinges on reducing in problem of decimation to ~~aa~~ a 1D ising model. Let us see this work in 2D, and then we will convert the English to equations for a general d.



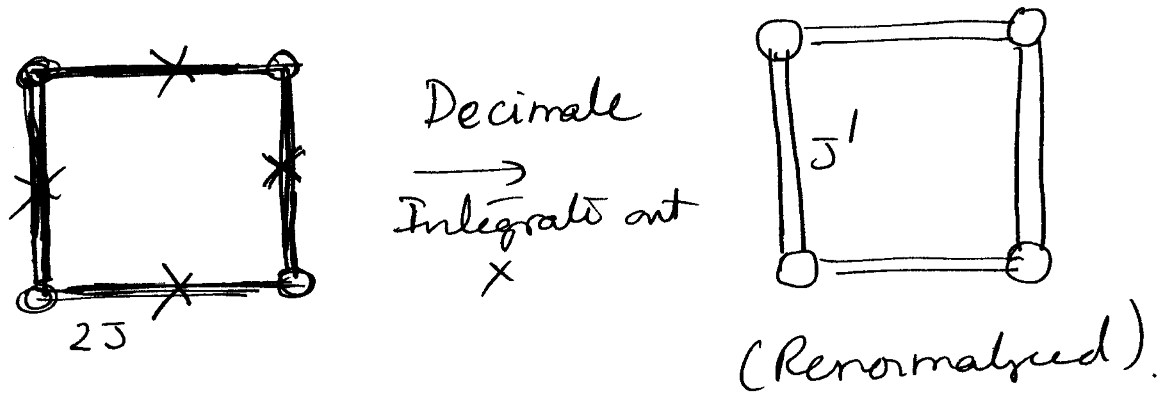
The key Migdal's idea is as follows: The key problem with the decimation procedure ~~tha~~ can be over come by the following trick. ~~Well see that~~ suppose the ☆ we want to find the effective Hamiltonian of the  plaquette (and must be of the simple ising form). ~~Note that presence of~~ The needs the integrating out of the × sites and ~~the~~ ☆ site. The start site is

a problem as it will produce "difficult to handle" interactions.
Migdal was brave, and he was Migdal!
So he decided to throw away(!!) the
☆ site! Into the trash can!



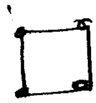
But to do this, we had ~~to throw away~~
do some thing with the bonds! This
Migdal did not throw into the trash,
instead he "stuck the bonds" to other
bonds. In other words,  the bonds
were stuck as shown, so each bond
became stronger! and of strength 2J.
~~[illegible]~~

Now, one can use the identical
formula as was one in 1D!

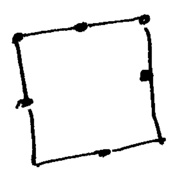


This is the idea! It is so very simple!

Let us then get down to the mathematics

Start with $\beta\mathcal{H}[J,h]$ on 

Next we "Migdalize" the lattice to obtain

$$\beta\mathcal{H}_m[J_m, h_m]$$ on 

As discussed

$$J_m = 2J$$

and $$h_m = h.$$ (This is done in such a way that each bond Hamiltonian of the Migdalized lattice looks like the 1-d system

$$\beta\mathcal{H}_m^{(ij)} = -J_m S_i S_j + \frac{h_m}{2}(S_i+S_j)$$

+ Const (drop this)

We know that the constant do not affect the R.G. flow, so we will not consider it at all. We now cast the variables into $t = e^{-2J}$ and $u = e^{-2h}$ . . mi.. The magnetized values are $t_m = t^2$ and $u_m = u$

Then

$$R_{S=2}[t, u] = \{t', u'\}$$

$$(t')^{-2} = \frac{(t^{-2}u^{-1} + t^2)(t^2 + t^{-2}u)}{(\sqrt{u} + \sqrt{u^{-1}})^2}$$

$$(u')^{-1} = \frac{(t^{-2}u^{-1} + t^2)}{(t^2 + t^{-2}u)}$$

We see, as expected, that if $u = 1$, then $u' = 1$ (Integrating out degrees of freedom do not produce a symmetry breaking field).

For $h = 0$ $(u = 1)$ we have the RG. transformation

$$(t')^{-2} = \frac{(t^{-2} + t^{2})^{2}}{4}.$$

For $h=0$, cone cleary obtain the following fixed pants

$t_0^* = 0$ (zero Temperate), $t_\infty^* = 1$ (∞ Temperate)

and another

$$t_c^* = \frac{1}{\sqrt{1 + \frac{1}{3}(54 - 6\sqrt{33})^{1/3} + \frac{(2(9+\sqrt{33}))^{1/3}}{3^{2/3}}}}.$$

$$\simeq 1.839$$

This correspons to

$J_c^* = 0.304\cancel{7}$ [Exact value 0.44]

~~For~~

For $t \approx 0$,

$$\frac{1}{t'^2} \simeq \frac{1}{4} \frac{1}{t^4} \quad \Rightarrow \quad t' = 2t^2$$

In other words $(\delta t') = 2(\delta t)^2$

We see that there is no ~~[illegible]~~ (δt) term and ~~hence~~ $\delta t' << \delta t$! Thus temperature is irrelevant at the P_0^* fixed point. Note this is quite different from $d=1$!

Moving now to P_∞^* fixed point where $(t_\infty^* = 1,\ h_\infty^* = 0)$, we see that

$$\delta t' \approx (\delta t)^2 \quad \text{again.}$$

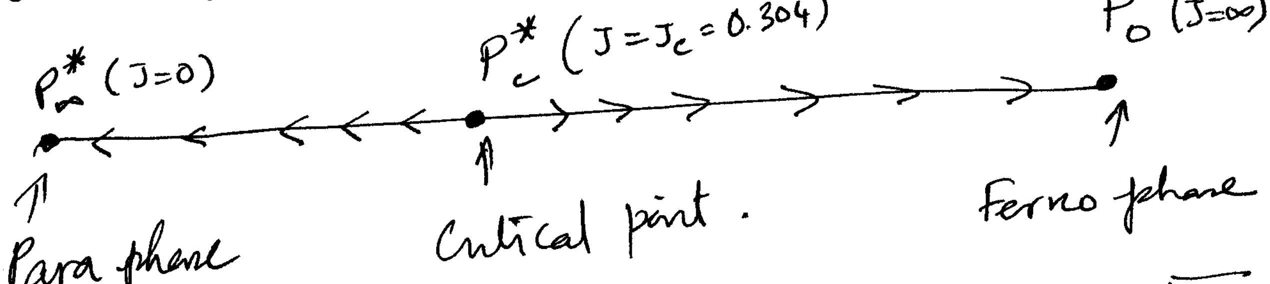
Thus temperature is irrelevant also at this fixed point

Now the most interesting ~~me~~ P_c^*, we can obtain (using mathematica) that

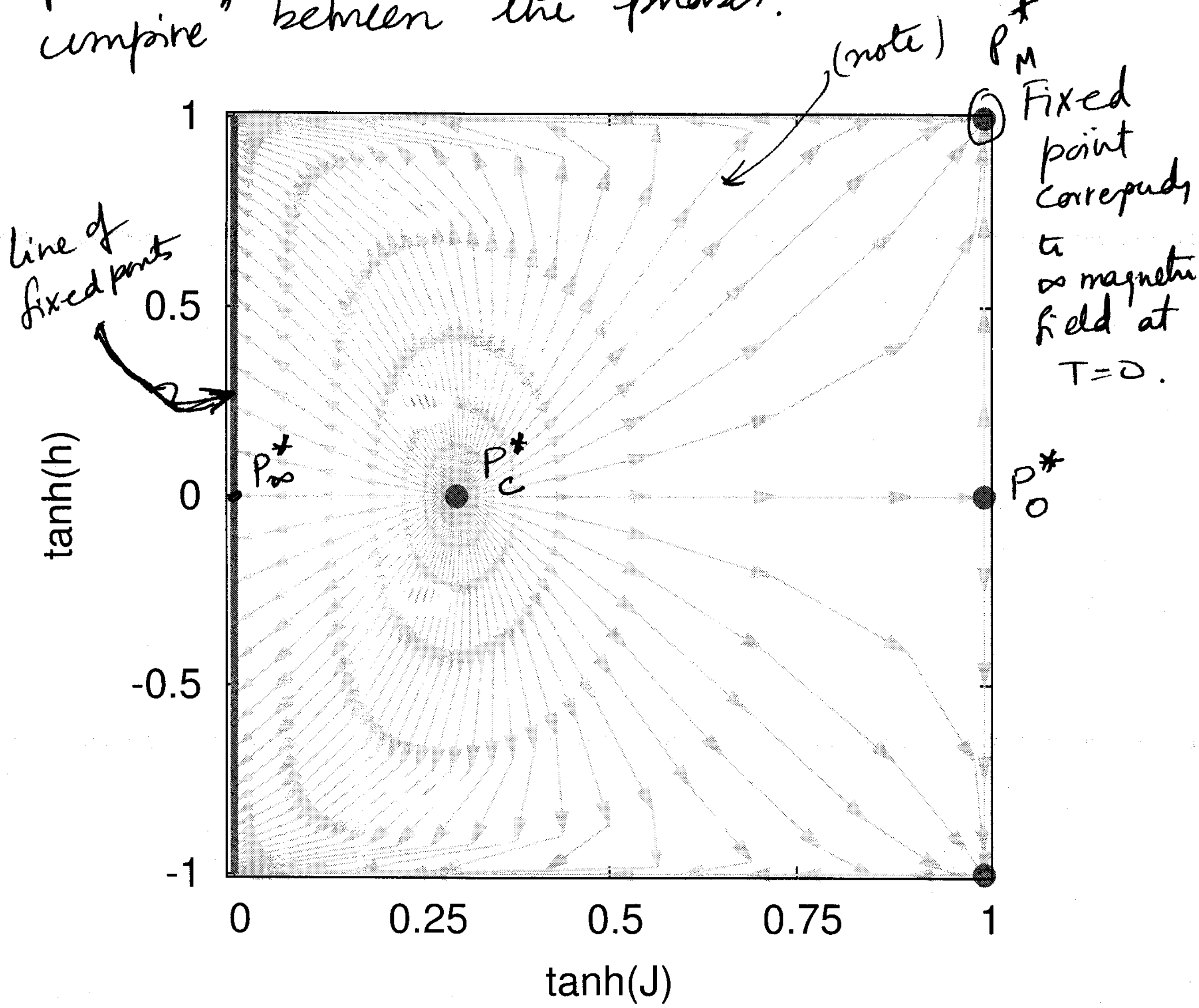
$$\delta t' = \lambda_t^c \, \delta t$$

$$\lambda_t^c = 1.678\cancel{5}7$$

Thus temperature is relevant at this fixed point. We can draw a flow diagram



· Note that at P_0^* and P_∞^*, there are no symmetry preserving relevant operators at these fixed points. They characterize the phases of the model, P_0^* characterizes the $T=0$ Ferro state, while P_∞^* characterizes the para phase. The critical point is the "neutral umpire" between the phases.



We can use the RG recurrence relations to obtain the full flow diagram of the 2D Ising model as shown above.

We can get a deeper understanding of the phase diagram by looking at various fixed points.

Lets start with P^*_0.

Here $\cancel{P^*_0}$ $\quad \delta t' = 2(\delta t)^2$ (we have seen this)

$$\delta \cancel{h}' = \cancel{\lambda}^0_h (\delta u)$$

$\lambda^0_h = 2 \Rightarrow \mu^0_h = 1.$, and the ~~operator~~ (to magnetic field h is a relevant operator at the fixed point P^*_0. This is not quite surprising as the energy of the state changes, in a magnetic field, and the flow takes it to the infinite magnetic field (marked by a circle) in the ~~figure~~ flow diagram. ξ

Now we move to the P^*_∞ ($t = 1$) regime (on the left side of the figure)

For $t = 1$, the recursion relation are

$$t' = 1 \quad \text{and} \quad (u')^{-1} = \frac{1}{u} \; !$$

Thus, again, we get a line of fixed points.

Now let us look at the the $t = 0$ line

Here the flow starts at P^*_σ fixed point and takes us away.

The recurrence relation is

$$(u')^{-1} = \frac{1}{u^2}$$

So any $u < 1$ will flow to P^*_M

P^*_M corresponds to $u^*_M = 0$, $t^*_M = 0$...

For $u = 0$, ~~what one~~ we see that the magnetic field is irrelevant at ~~the~~ P^*_M. On the other hand ~~the~~ ~~temp~~ temperature is highly relevant, in fact if $t \neq 0$, and $u = 0$,

$$t' = 1 \left(\lim_{u \to 0} \frac{(t^{-2}u^{-1})t^2}{u^{-1}} \right)$$

$$= 1 !$$

So the flow instantly takes us to the fixed point $(t = 1, u = 0)$.

We can understand the flows. All lines originate at a fixed point along a relevant direction and terminate at a fixed point along an irrelevant direction

Note that certain flows show some interesting behaviour. See the line indicated by "note". Starting from P_c^* it flows towards the P_{mag}^* fixed point dominated by the magnetic field. As soon as it approach close to P_m^*, the strongly relevant ~~direction~~ operator (temperature) at this ~~of dimen~~ fixed point comes to operation and the flow "turns" or crosses over to the ~~x~~ ∞ temperature fixed point. In other words each point on the flow diagram typically "belongs to" a particular fixed point, i.e, belongs to the "basin of attraction" of the fixed point. The properties of the system ~~with~~ with those values of coupling constants will be determined by that fixed point. When the ~~fixpoint~~ parameters are close to say ~~is close to~~

P_0^*, one "sees" the system behaving like a magnet! The RG picture therefore provides a remarkably simple, yet powerful picture of the phase diagram of the Ising magnet..

Finally let us calculate the critical exponent at the critical fixed point P_c^*. Here

$$\delta t' = \lambda_t^c \, \delta t$$

$$\delta \cancel{\mu}' = \lambda_h^c \, \delta u$$

$$\lambda_t^c = 1.67857 = 2^{\mu_t^c} \Rightarrow \mu_t^c = 0.747$$

$$\lambda_h^c = 1.83929 = 2^{\mu_h^c} \Rightarrow \mu_h = 0879$$

(Drop the c subscript)

Noting that $f(s^{\mu_t^c}\delta t, s^{\mu_h^c}\delta u) = s^{d} f(\delta t, \delta u)$

we get

$$\alpha = 2\left(1 - \frac{1}{\mu_t}\right) = -0.677$$

$$\beta = \frac{1 - \mu_n/2}{\mu_t/2} = 1.5$$

$$\gamma = \frac{\mu_n - 1}{\mu_t/2} = 0.647$$

$$\delta = \frac{1}{\left(\frac{2}{\mu_n} - 1\right)} = 0.784$$

$$\nu = \frac{2-\alpha}{2} = \frac{1}{\mu_t} = 1.338$$

$$\eta = 2 - \frac{\gamma}{\nu} = 2 - \frac{\mu_t 2[\mu_n - 1]}{\mu_t}$$

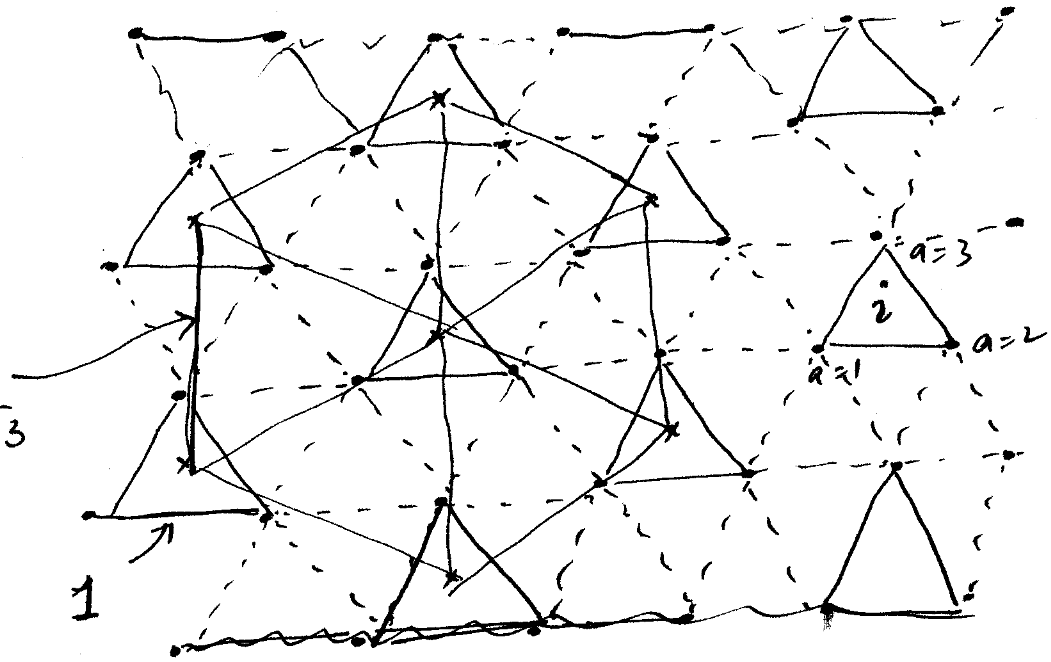
$$= 4 - 2\mu_n$$

$$= 2.24.$$

what is quite remarkable is that the theory does predict non Landau Ginzburg critical exponent and also produces a non zero value of η. The answers are quite off from the correct ones. This could be because of the drastic approximations made by Migdal. While there are justifications for this, it is clear that one needs a more systematic and controlled technique.

Such a technique was developed by later authors who used both Wilson's and Kadanoff's block spin ideas. Block spin is that during coarse graining one replaces the spin in a block by the majority — the majority rule. We will see this next.

Coarse graining (block spin) RG transformation of a triangular lattice.



$\beta\mathcal{H} = \qquad\qquad\qquad = \beta\mathcal{H}[s_a^i]$

$\beta H = -J \sum'_{\substack{\langle i,j\rangle \\ a,b}} s_a^i s_b^j \quad - h \sum'_{i,a} s_a^i$

New degree of freedom in each triangle σ^i

$\sigma^i = \text{Sign}\left[\sum'_a s_a^i\right] = C[s_a^i]$

these are Kadanoff's block spins.

Idea

$$e^{-\beta'\mathcal{H}'[\sigma^i]} = \sum'_{\{s_a^i\}} \prod_i \delta(\sigma^i - C[s_a^i]) \, e^{-\beta\mathcal{H}[s_a^i]}$$

This looks like a hard problem! And it is!

A clever way (inspired by Wilson) was found by Neimeyer and van Leeuwen (NVL)

(See PRL, 32, 1411 (1973)). These authors decided to write

$$\beta\mathcal{H}[s_a^i] = \beta\mathcal{H}^0[s_a^i] + \beta\mathcal{H}^I[s_a^i]$$

$\beta\mathcal{H}^h[s_a^i]$

$\beta\mathcal{H}^0$ is the term corresponding to unconnected triangles ~~as~~ with solid lines. The terms with dashed lines is $\beta\mathcal{H}^I$. In other words one is viewing the triangular lattice as made up of a triangular basis, of three sites with $a=1,2,3$. The ~~in~~ intra basis interaction Hamiltonian is $\mathcal{H}_0$ and the interaction between different basis units is $\beta\mathcal{H}^I$. We shall also take the magnetic part of the Hamiltonian into the perturbation.

Let us call

$$\beta\mathcal{H} = \beta\mathcal{H}^0 + \beta V$$

The

$$\beta\mathcal{H}' = -\ln\left[\sum_{s_a^i} \prod_i \delta(\sigma^i - C[s_a^i])\, e^{-(\beta\mathcal{H}+\beta V)}\right].$$

Now

$$\sum_{s_a^i}' e^{-\beta \mathcal{H}_0} \left[\prod_i (\sigma^i - C[s_a^i]) \; e^{-\beta V} \right]$$

$$= Z_0 \sum_{\{s_a^i\}}' \left\langle \prod_i (\sigma^i - C[s_a^i]) \left(1 - \beta V + \frac{(\beta V)^2}{2}\right) \right\rangle_0$$

We can formally treat this as a series in V.

We need the log of this quality

$$= -\ln Z' = (\beta \mathcal{H}')(\sigma^i) \quad C$$

$$= -\ln Z_0 - \ln\left(\sum_{s_a^i}' \left\langle \prod_i \overline{(\sigma^i - C[s_a^i])} \left(1 - \beta V + \frac{(\beta V)^2}{2} \cdots \right) \right\rangle_0 \right.$$

Recall $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} \cdots$

$$= \cancel{\langle -\beta V \rangle_0 + \beta V} \quad - \langle \beta V C \rangle_0 + \frac{1}{2} \langle C (\beta V)^2 \rangle_0 - \frac{1}{2} \langle \beta V \rangle^2 + \cdots$$

$$= - \langle \beta V C \rangle_0 + \frac{1}{2}\left[\langle C (\beta V)^2 \rangle_0 - \langle C \beta V \rangle^2 \right] + \cdots$$

Thus apart from constants

$$\beta \mathcal{H}' = \langle \beta V C \rangle_0 - \frac{1}{2}\left[\langle C(\beta V)^2 \rangle_0 - \langle C \beta V \rangle_0^2 \right]$$

We will simply stop at the first order in our discussion.

$$\beta H' = \langle \beta V \rangle_0$$

Now recall that

$$\beta V = -h \sum_i{}' \sum_a{}' s_a^i$$
$$- J \sum_{\substack{\langle ij \rangle \\ (i \neq j)}}{}' s_{(a)}^i s_{(b)}^j$$

appropriate ones

↳ nearest neighbours

Let us look at term by term.

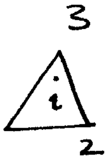
$$\beta H' = - h \sum_i{}' \sum_a{}' \langle s_a^i \, \delta(\sigma^i - C[s_a^i]) \rangle_0$$

$$- J \sum_{\substack{ij \\ (\text{near nghbrs})}}{}' \langle s_{(a)}^i \delta(\sigma^i - C[s_a^i]) \rangle_0 \langle s_{(b)}^j - \delta(\sigma^j - C[s_b^j]) \rangle_0$$

3

The last ~~in~~ equality works because in $\beta\mathcal{H}^0$, i ~~do~~ basis does not talk to the ~~+~~ j basis.

What we need to do is calculate $\langle s_a^i \, \delta(\sigma^i - C[s_a^i]) \rangle_0$. Since βH^0 does ~~not~~ has symmetry in the a index, we can calculate any ~~+~~ one of them.

$C[s_1^i + s_2^i + s_3^i] = \mathrm{sign}[s_1^i + s_2^i + s_3^i]$ 

σ^i	s_1^i	s_2^i	s_3^i	Energy	Weight
↑	↑	↑	↑	$-3J$	e^{3J}
	↓	↑	↑	J	e^{-J}
	↑	↓	↑	J	e^{-J}
	↑	↑	↓	J	e^{-J}
↓	↓	↓	↓	$-3J$	e^{3J}
	↑	↓	↓	J	e^{-J}
	↓	↑	↓	J	e^{-J}
	↓	↓	↑	J	e^{-J}

4

$$\langle s^i_a \, \delta(\sigma^i - c[s^i_a]) \rangle$$

$$= \frac{e^{3J} + e^{-J}}{e^{3J} + 3e^{-J}} \quad \text{if } \sigma^i = \uparrow$$

$$= -\frac{(e^{3J} + e^{-J})}{e^{3J} + 3e^{-J}} \quad \text{if } \sigma^i = \downarrow .$$

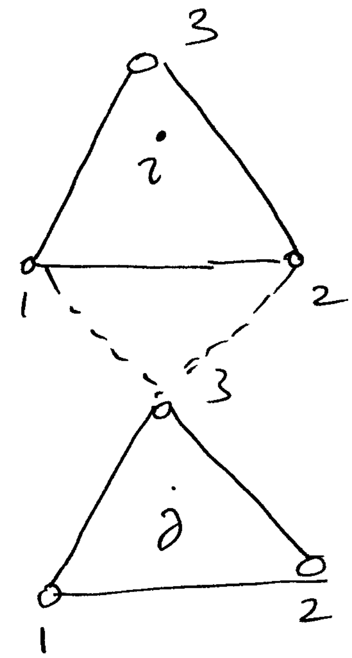
$$= \left(\frac{e^{3J} + e^{-J}}{e^{3J} + 3e^{-J}} \right) \sigma^i$$

Let us ~~nee~~ look at the magnetic term first,

$$\langle (\beta V)^h_c \rangle = \underbrace{-3h \left(\frac{e^{3J} + e^{-J}}{e^{3J} + 3e^{-J}} \right)}_{h'} \sum_i{}' \sigma^i$$

$h' \leftarrow$ new (renormalized) magnetic field.

Moving over to the interaction term, we need to look more closely at the geometry.



The connections between the (i,j) basis are shown in the figure.

The term is

$$\left\langle -J\left(s_1^i s_3^j + s_2^i s_3^j\right)\delta(\sigma^i - c[s_\alpha^i])\,\delta(\sigma^j - c[s_\alpha^i])\right\rangle$$

$$= -J\left(\frac{e^{3J}+e^{-J}}{e^{3J}+3e^{-J}}\right)^2\left[\sigma^i\sigma^j+\sigma^i\sigma^j\right]$$

$$= \underbrace{-2J\left[\frac{e^{3J}+e^{-J}}{e^{3J}+3e^{-J}}\right]^2}_{J'}\sigma^i\sigma^j$$

$$\beta\mathcal{H}' = -J'\sum_{ij}{}'\,\sigma^i\sigma^j - h'\sum_i{}'\,\sigma^i$$

6

Thus

$$R_{s=\sqrt{3}}\,[J, h] \overset{\text{perturbative in } h}{=}$$

$$\left[2J \left[\frac{e^{3J} + e^{-J}}{e^{3J} + 3e^{-J}} \right]^2 , \underset{\text{"small" } h.}{3h} \left[\frac{e^{3J} + e^{-J}}{e^{3J} + 3e^{-J}} \right] \right]$$

We can obtain the fixed points

$h^* = 0$ and

$$J^* = 2J^* \left[\frac{e^{3J^*} + e^{-J^*}}{e^{3J^*} + 3e^{-J^*}} \right]^2$$

Fixed points are

$J_0^* = 0$, $J_\infty^* = \infty$ and

$J_c^* = \frac{1}{4} \ln \left[1 + 2\sqrt{2} \right] = 0.3356$.

(Exact value is about 0.27)

The two fixed ~~[illegible]~~ T_c^*. ~~[illegible]~~ to the $T = \infty$ point $J_0^* = 0$ corresponds, and J_∞^* fixed point is the paramagnet,

$T=0$ ~~for~~ phase. The interesting fixed point is J_c^*

Let us find $L_{S=\sqrt{3}}[P_c^*]$. This is best done using mathematica

$$\begin{aligned} \delta J' &= \\ \delta h' &= \end{aligned} \begin{bmatrix} \lambda_J & 0 \\ 0 & \lambda_h \end{bmatrix} \begin{bmatrix} \delta J \\ \delta h \end{bmatrix} \quad \begin{cases} \delta J = J - J_c^{*} \\ \delta h = h \end{cases}$$

$$\lambda_J = \frac{2\left(10 + 7\sqrt{2} + (5+3\sqrt{2})\log(1+2\sqrt{2})\right)}{(2+\sqrt{2})^2}$$

$$\lambda_h = \frac{3}{\sqrt{2}}.$$

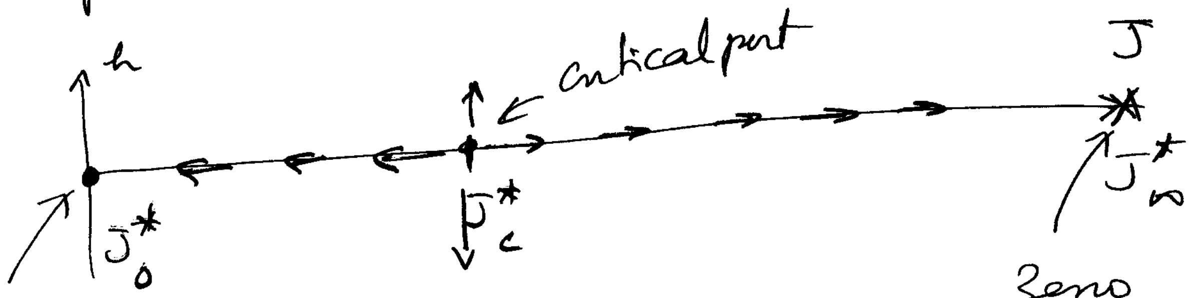
We can now obtain $\lambda_J = S^{\mu_t} = \sqrt{3}^{\mu_t}$

$$\mu_t = \frac{\log[\lambda_J]}{\log[\sqrt{3}]} = 0.882$$

$$\mu_h = \frac{\log[\lambda_h]}{\log(\sqrt{3})} = 1.369$$

8

Since both μ_t and μ_h are > 0, both are relevant operators. Based on this we can now put down a phase diagram



Infinite temperature paramagnet.

Zero temperature

For the sake of completeness, let us put down calculated values of the critical exponents. ~~and~~ at the critical part and ~~calc~~ compare them with exact results. (note $f(s^{\mu_t}t, s^{\mu_h}h) = s^2 f(t,h)$)

$$\alpha = 2 - \frac{2}{\mu_t} = -0.267 \quad \text{(Exact is log divergence)}$$

$$\beta = \frac{1 - \mu_h/2}{\mu_t/2} = 0.715 \quad \left[\begin{matrix}\text{Exact} \\ 1/8\end{matrix}\right]$$

$$\gamma = \frac{\mu_h - 1}{\mu_{t}/2} = 0.83\cancel{7} \quad (\text{Exact } 7/4)$$

$$\delta = \frac{1}{\frac{d}{\mu_h} - 1} = 2.169 \quad (\text{Exact } 15)$$

$$\nu = \frac{1}{\mu_t} = 1.133 \quad (\text{Exact } 1)$$

$$\eta = 2 - \frac{\gamma}{\nu} = 1.26 \quad (\text{Exact} = 1/4)$$

We see that the results are way off! ~~Our~~ The scheme used here is way too approximate; but illustrates the concepts and obtains non Landau-Ginzburg critical exponents. A more detailed calculation including $\langle(\beta V)^2\rangle$ terms etc gives much better results, see Neimeijer and van Leeuwen's paper.

10 | 60

We are now faced with an interesting situation. We know that the Renormalization group idea, or more precisely the idea that the critical point is scale invariant, works. It produces "non-classical" or non-Landau-Ginsburg exponents. And yet the situation is far from satisfactory since there is no "framework" for doing calculation. ~~Mido~~ Migdal[2]-Kadanoff approach and even ~~Wei~~ Neimeijer-Van Leeuwen one based on ad hoc approximations (even ~~of~~ while ~~Neime~~ N-vL is post Wilsonian) and it is not clear how to improve them (thats why ad hoc). Wilson, in the late 60s, inspired by much of this work and his own numerical work, and field theory background, found this frame work. This also unified critical phenomena with problems of high energy physics.

Wilson's Momentum Shell RG.

A key difficulty faced in the Ising like model was the enforcement of Ising like variable, i.e, after the R_s transformation we want the new variables also to be Ising spins (with a renormalized Ising Hamiltonian). Wilson realized that it ~~it~~ is better to give up this approach and adopt a more general approach based on symmetry arguments. This is precisely the Landau-Ginsburg ~~ap~~ framework. Focusing on O(n) symmetry

We can write a Landau-Ginzburg Hamiltonian as

$$\mathcal{H} = \int d^d r \, \frac{1}{2} \sum_i{}' (\vec{\nabla} \phi_i)^2 + \frac{t}{2} \sum_i{}' \phi_i^2(\vec{r}) + \frac{u}{4} \Big(\sum_i{}' \phi_i^2(\vec{r}) \Big)^2$$

$$+ \sum_{m_k, m_\phi}{}' u_{2m_k, 2m_\phi} \left((\nabla)^{2m_k} \phi^{2m_\phi} \right)$$

We have written also a generic ~~term~~ term with $2m_k$ derivatives, and $2m_\phi$ ϕ variables, in such a way that all terms have O(n) symmetry. Note $u_{2m_k=0,\, m_\phi=2} = u$

How do we generate a R_S for this?
Note that there is always an ultraviolet cut off Λ in momentum space. Wilson ~~considered local action of the fi~~ ~~construed~~ the cast this in momentum space
Taking V to be volume of our d-dimensional box and defining

$$\phi_i(\vec{r}) = \frac{1}{\sqrt{V}} \sum_{k}' e^{i\vec{k}\cdot\vec{r}} \phi_i(\vec{k})$$

we get,

$$\mathcal{H}[\vec{\Phi}(\vec{k})] = \sum_{|\vec{k}|<\Lambda}' \frac{1}{2}(k^2+t)|\vec{\Phi}(k)|^2 \quad - \underbrace{h_i}_{\text{magnetic field}} \frac{\phi_i(0)}{\sqrt{V}}$$

$$+ \frac{u}{4V^2} \underbrace{V}_{\text{comes from } \delta \text{ momentum conserving } \delta \text{ function}} \cdot \sum_{\substack{i,j\\k,k'\\q}}' \phi_i(\tfrac{q}{2}+\vec{k})\, \phi_j(-\tfrac{q}{2}+\vec{k}')\, \phi_j(-\tfrac{q}{2}-\vec{k}')\, \phi_i(\tfrac{q}{2}-\vec{k})$$

$$+ \frac{u_{2m_k, 2m_\varphi}}{V^{m_\varphi}} V \sum' \cdots$$

We should now note that there is a small change in our way of writing LG ~~H~~ Hamiltonian. We no-longer write $K(\nabla\phi)^2$ but simply $(\nabla\phi)^2$. We ~~must~~ have thus chosen a different set of units.

Let us do a more careful dimensional analysis

$$[\Lambda] = L^{-1} \qquad [r] = L$$

Since $\mathcal{H}$ is dimensionless, we must have

$$\frac{1}{L^2}[\phi(r)]^2 L^d = L^0$$

$$\Rightarrow \quad [\phi(\vec{r})] = L^{\left(\frac{2-d}{2}\right)}$$

It now follows that

$$[\phi(\vec{k})] = L^{\cancel{1}}, \quad [t] = L^{-2}$$

$$[u] = L^{(\cancel{d}-\cancel{4})} \; ; \quad [h] = L^{-\left(\frac{d}{2}+\cancel{1}\right)}$$

~~Armed with this understanding, we will now go through Wilson's procedure.~~

Generically

$$\frac{u_{2m_k,2m_\varphi}}{V^{m_\varphi}} \; V \; \overset{\cancel{\text{will be}}}{} \; k^{2m_k} \phi^{2m_\varphi}$$

~~dim~~ will be dimmensionless

$$u_{2m_k,2m_\varphi} = L^{d(m_\varphi-1)+2m_k-2m_\varphi.}$$

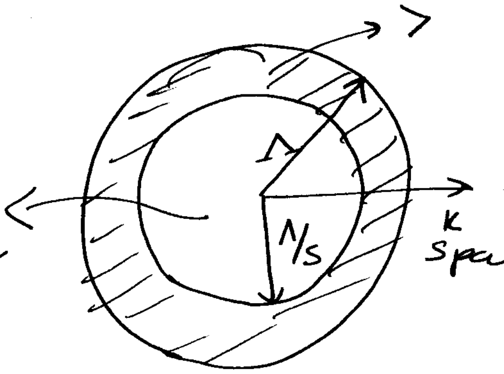
$$= L^{\cancel{d(m_\varphi)}\,(d-2)m_\varphi+2m_k-d}$$

$$\Rightarrow \quad [u_{2m_k,2m_\varphi}] = L^{(d-2)m_\varphi+2m_k-d}$$

Armed with the dimensional understanding, let us go through Wilson's approach. Wilson constructed R_s in three steps.

① $\boxed{\text{Integrate}}$: In this steps all the "modes" or fields with momenta (called $\phi^>(\vec{k})$) are integrated between $\frac{\Lambda}{s}$ and Λ, out to obtain an effective ~~actio~~ Hamiltonian for fields in $0<|\vec{k}|<\frac{\Lambda}{s}$.

(called $\phi^<(\vec{k})$)



$$e^{-\beta\mathcal{H}^{\text{eff}}[\phi^<(k)]} = \int \mathcal{D}[\phi^>(k)]\, e^{-\beta\mathcal{H}[\phi^<,\phi^>]}$$

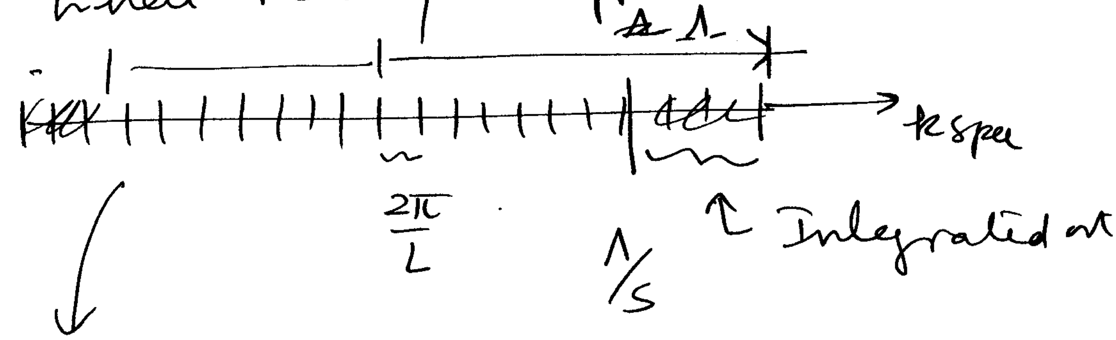
② $\boxed{\text{Rescale}}$: The ~~fo~~ next step is to rescale real space and associated momentum space

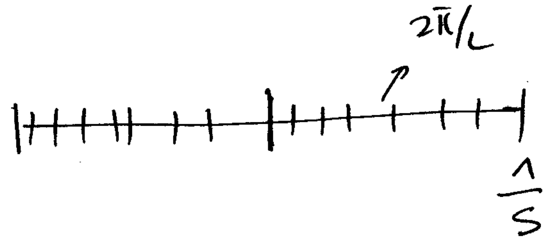
$$\vec{r}' = \frac{\vec{r}}{s} \quad \text{or} \quad \vec{k}' = s\vec{k}$$

Thus $\Lambda' = s\cdot\left(\frac{\Lambda}{s}\right) = \Lambda$.

This restores the cutoff momentum back to Λ

This is what really happens ~~as~~ as illustrated in 1d.

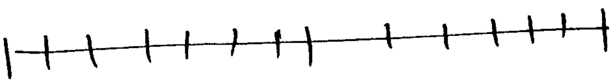




$\Delta k = \frac{2\pi}{L}$

make $k' = sk$ or $\Delta k' = \frac{2\pi s}{L} = \frac{2\pi}{L'}$

→ "Less" density of points



(but both are equal since $L = L' = \infty$!)

(3) Renormalize. now we need to ~~red~~ redefine the fields $\varphi(\vec{k})$ (or equivalently ~~f~~ $\varphi(r)$) such that the Hamiltonian ~~is~~ is same. the form of the Hamiltonian is same. The key thing here is that the coefficient of the $|\nabla\varphi|^2$ term is $\frac{1}{2}$, and we want to retain this to be $1/2$. (Clearly this is not the only way to do this; and we will see what this particular choice means later!)

The last rent can be written as

$$\phi_i'(k') = \zeta^{-1}(s)\, \phi_i^{<}(k)$$

$\uparrow$ Renormalization of field.

~~This We~~
These ~~these~~ steps will produce

$$\beta\mathcal{H}'[\phi'] = \sum_{k'}' \Big\{ \frac{1}{2}\big((k')^2 + t'(s)\big)|\phi'(k')|^2$$

$$+ \frac{u'(s)}{V'} \sum_{q',k',k}' \phi_i\Big(\frac{q'}{2}+k'\Big) \dots \text{etc}\Big)$$

~~This then~~ $$+ \frac{u'_{2m_k 2m_q}(s)}{(V')^{m_q}} \dots$$

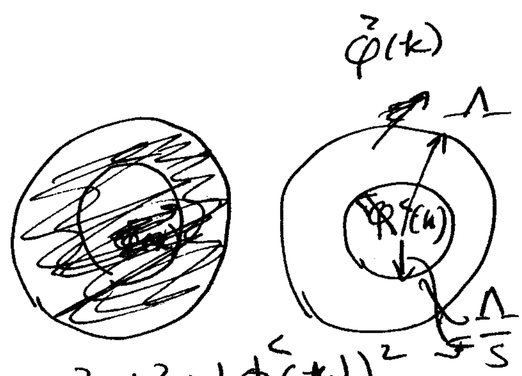
This produces the RG flow

$$R_s\big[t, u, u_{2m_k, 2m_q} \dots\big]$$
$$= \big\{ t'(s), u'(s), u_{2m_k, 2m_q}(s) \big\}.$$

Let us see this in action. The simplest place to start is the so called Gaussian model. Here $t \neq 0$, but all us are zero.

$$\beta H[\phi_k] = \sum_k{}' \frac{1}{2}(k^2+t)\,|\phi(\vec{k})|^2.$$

(Gaussian model).



① Integrate

$$Z = \prod\int d\{\phi^<(k)\}\, e^{-\sum'^{<}_{|k|\leq\frac{\Lambda}{s}} \frac{1}{2}(k^2+t^2)|\phi^<(k)|^2}$$
$$\times \prod\int d\{\phi^>(k)\}\, e^{\sum'^{\Lambda}_{\Lambda/s} \frac{1}{2}(k^2+t)|\phi^>(k)|}$$

Fields with $k<\frac{\Lambda}{s}$ are called $\varphi^<(k)$ and fields with $\frac{\Lambda}{s}<k<\Lambda$ are called $\varphi^>(k)$.

We get

$$\beta H^{eff}[\phi^<(k)] = \sum_{k<\frac{\Lambda}{s}}{}' \frac{1}{2}(k^2+t)|\phi^<(k)|^2$$
$$+ \text{~~Const~~ Constant}$$

② Rescale chge $k' = sk$.

$$\beta H^{eff}[\phi^<(k)] = \sum_{|k'|<\Lambda}{}' \frac{1}{2}\left(\frac{k'^2}{s^2}+t\right)|\phi^<(k)|^2$$

~~Write~~ (3) Renormalize:

Write: $\phi^{<}(\vec{k}) = \zeta(s)\, \phi'(\vec{k}')$

we get

$$\beta H'[\phi'] = \sum_{|\vec{k}'|<\Lambda}' \frac{1}{2}\left(\frac{(\vec{k}')^2}{s^2} + t\right) \zeta^2(s)\, |\vec{\phi}'(\vec{k}')|^2$$

Choosing $\zeta^2(s) = s^2$ i.e, $\zeta(s) = s$,

we get

$$\beta H'[\phi'] = \sum_{|\vec{k}'|<\Lambda}' \frac{1}{2}\left((\vec{k}')^2 + s^2 t\right) |\vec{\phi}'(\vec{k}')|^2$$

which is precisely in the form as we started with but with new coupling certain. We say

$$R_s\left[\{t,\ u=0,\ u_{2m_\phi,2m_k}=0\}\right]$$

$$= \{s^2 t,\ u=0,\ u_{2m_\phi,2m_k}=0\}.$$

We have thus generated the RG flow (or recurrence) relation. Before we proceed to analyze this RG flow, we ~~would take~~ will analyze the ~~map~~ meaning of this.

We ~~cannot~~ ask "why" did we get the RG recurrence of the kind we got?

Note that in this Gaussian model, the $\phi^<(k)$ modes are unaffected by the low energy modes. ~~Thus~~ For this, rescaling and renormalization is completely determined by dimensional considerations

For example

$$[t] = L^{-2} \text{ and } [\phi(k)] = L$$

Thus scaling $r' = \frac{r}{s}$, means that

$$t' \to s^2 t \quad \text{and} \quad \phi'(k') \to s^{-1}\phi(k)$$

simply by dimensional considerations!

If $\phi'(k') = s^{-2+\frac{\eta}{2}} \phi(k)$, then

we say that the field has an anomalous dimension, i.e., it scales differently from what is obtained from naive dimensional analysis. In this Gaussian example $\eta = 0$, and everything follows dimensional ~~and~~ analysis.

Let us ~~star~~ come back to the RG ~~Rel~~ relations and study the fixed points.

If we had also a magnetic field h, then

$$h' = s^{+(\frac{d}{2}+1)} h .$$

Thus

$$R_s\left[\{t, u=0, u_{2m}=0, \tilde{c}_{m\phi}; h\}\right]$$

$$= \left[s^2 t, u=0, u_{2m}, c_{m\phi}=0, s^{(\frac{d}{2}+1)} h\right]$$

The fixed point

$$p^*_G = \{0, 0, \ldots 0\} \text{ simple}.$$

$$t^*_G = 0 : h^*_G = 0 \ldots$$

Note

$$f_s(t, h) = \bar{s}^{d} f(s^{\overbrace{2}^{\mu_t}} t, s^{\overbrace{(\frac{d}{2}+1)}^{\mu_h}} h)$$

free energy density (singular)

or $f(t, 0) = t^{+\frac{d}{2}}$ ~~$f(1,*)$~~

$$\Rightarrow \alpha = 2 - \frac{d}{2} = \frac{4-d}{2}.$$

~~(not quite meaningful)~~

~~(Since $t<0$ has a problem)~~

If we use the hyperscaling relation we see that

$$\xi^{-d} \sim t^{d/2} \Rightarrow \xi = t^{1/2}$$

or $\nu = 1/2$.

Thus we see that α and ν are exactly same as that obtained within the Gaussian fluctuation theory. ~~#~~ The anomalous dimension $\eta = 0$.

The R.G recursion relations can be cast as a flow equation. For this put $s = e^{l}$

Now

$$s\frac{d}{ds} = \frac{d}{dl}$$

$$\Rightarrow \left.\begin{aligned} \frac{dt}{dl} &= 2t \\ \frac{dh}{dl} &= \left(\tfrac{d}{2}+1\right)h \end{aligned}\right\} \beta(t,h)$$

Fixed point $\beta(t,h) = 0$ ~~gives~~ gives the gaussian fixed points. Note that in this language the coefficient of the linear term are the exponents μ_t and μ_h.

We also see the both t and h are relevant operators at ~~t=0~~ P_G^*

We see that the Gaussian fixed point has quite similar physics as the Gaussian approximation (saddle point) of the GL theory. So this <u>cannot</u> be the real story!

What is the way out? Well we showed at at P_G^*, all u^*s are zero. The question is are they all irrelevant? How do we answer this question?

There is a cheap way to check if you are right. We know that one key contribution to renormalized coupling constants comes from their dimensions. For example, we see that

$$u' = s^{4-d} u$$

since $[u] = L^{(d-4)}$ and

$$u_{2m_k, 2m_\phi} = s^{-((d-2)m_\phi + 2m_k - d)} u_{2m_k, 2m_\phi}$$

Both of these are results of dimensional analysis.

One can immediately write out the leading contribution to the β function as

$$\frac{du}{dl} = (4-d)\,u$$

$$\frac{d\,u_{2m_k,2m_\varphi}}{dl} = -\left((d-2)\,m_\varphi + 2m_k - d\right) u_{2m_k,\,2m_\varphi}.$$

~~Let's look at d=3 (for example)~~

m_k ~~these~~ For any $d<4$, u is a relevant operator! ~~Thus~~ if there is even a small u, we will flow ~~away~~ away from the Gaussian fixed point! And for $d>4$, we see that it is irrelevant.

What about ~~d~~ other operators, we see that any ∇ terms ~~make~~ takes things towards irrelevant

So look at $(d-2)\,m_\varphi - d < 0$

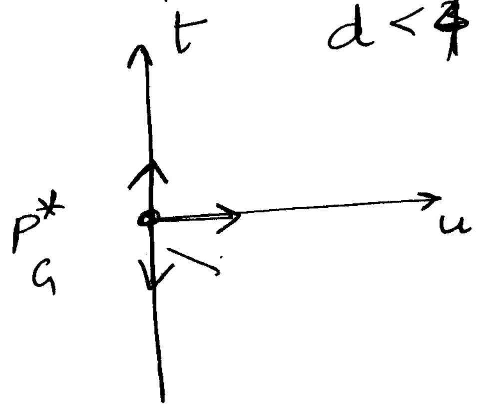
$$\Rightarrow \quad m_\varphi < \frac{d}{d-2}$$

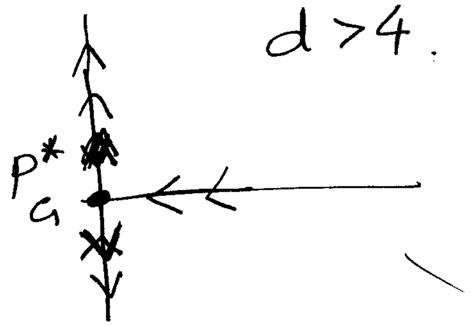
In $d=3$ $\quad m_\varphi < \frac{3}{1}$ only possible value is 2! (which is u).

So all operators are irrelevant ~~in~~ with $m_\varphi \geq 3$. in $(d \geq 3)$ are totally irrelevant.
So we need to worry only about u.

What have we learnt? Within a simple RG procedure (dimensional) arguments, we concluded that P_G^* $(t=0, u=0)$ is an unstable fixed point in $d<4$ and a stable one in $d>4$.
All other operators are irrelevant.

The picture is





Note that the results also ~~explain~~ explain why ~~Ga~~ Saddle point exponents are ~~made~~ exact for $d > d_c^u$! ~~So all in d~~ All "interaction" terms are irrelevant for $d > d_c^u$; and hence do not affect the critical behavior... So we should focus on ~~...~~ ~~the~~ The key question ~~So~~ $d<4$, is where does the flow along u axis take us? To another fixed point? We have to explore.

It turns out that there is a key difficulty in the construction of R_S. If we are near the Gaussian fixed point, then u is "small", and one might like to use perturbation theory. For this we write (we drop $\beta\mathcal{H}$ and simply write $\mathcal{H}$)

$$\mathcal{H} = \underbrace{\sum_{|\vec{k}|<\Lambda}' \frac{1}{2}(k^2+t)\,|\phi(k)|^2}_{\mathcal{H}_0} + \underbrace{\frac{u}{V}\sum_{k,k',q}' \phi_i\!\left(\frac{q}{2}+k\right)\phi_j\!\left(\frac{q}{2}-k\right)\phi_j\!\left(-\frac{q}{2}-k'\right)\phi_i\!\left(-\frac{q}{2}+k'\right)}_{H_u}.$$

Using the result

$$Z = \int \mathcal{D}[\varphi]\, e^{-(\mathcal{H}_0+\mathcal{H}_u)}$$

$$\frac{Z}{Z_0} = \left\langle e^{-\mathcal{H}_u}\right\rangle_0 = \sum_m' \frac{(-)^m}{m!}\left\langle (\mathcal{H}_u)^m\right\rangle_0 .$$

Supposing that we are in the disordered

phase, we can calculate the susceptibility

$$\chi(k) = \langle \phi_i(k)\, \phi_i(-k) \rangle \quad \text{(no sum on } i\text{)}.$$

$$= \frac{\langle \phi_i(k)\, \phi_i(-k)\, e^{-\mathcal{H}_u} \rangle_0}{(Z/Z_0)}$$

$$= \frac{\langle \phi_i(k)\, \phi_i(k)\, e^{-\mathcal{H}_u} \rangle_0}{\langle e^{-\mathcal{H}_u} \rangle_0}.$$

We will (very briefly) review how calculations such as the series for $\frac{Z}{Z_0}$ are carried out.

$$\mathcal{H}_u = \frac{u}{4V} \sum_{k_1' k_2' k_3' k_4'}' \phi_i(k_1')\, \phi_j(k_2')\, \phi_j(k_3')\, \phi_i(k_4')\, \delta_{k_1'+k_2'+k_3'+k_4',0}.$$

Now $\langle \phi_i(k_1')\, \phi_j(k_2')\, \phi_j(k_3')\, \phi_i(k_4') \rangle_0$ is obtained by something called the "Wick's theorem". The essential content of Wick's theorem is that every higher moment of a Gaussian distribution can be obtained from the second moment

For example

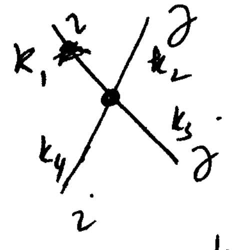
$$\langle \phi_i(k_1')\, \phi_j(k_2')\, \phi_j(k_3')\, \phi_i(k_4') \rangle_0$$

$$= \langle \phi_i(k_i')\, \phi_j(k_2') \rangle_0 + \langle \phi_i(k_4')\, \phi_j(k_3') \rangle_0 \langle \phi_j(k_2')\, \phi_i(k_4') \rangle_0$$

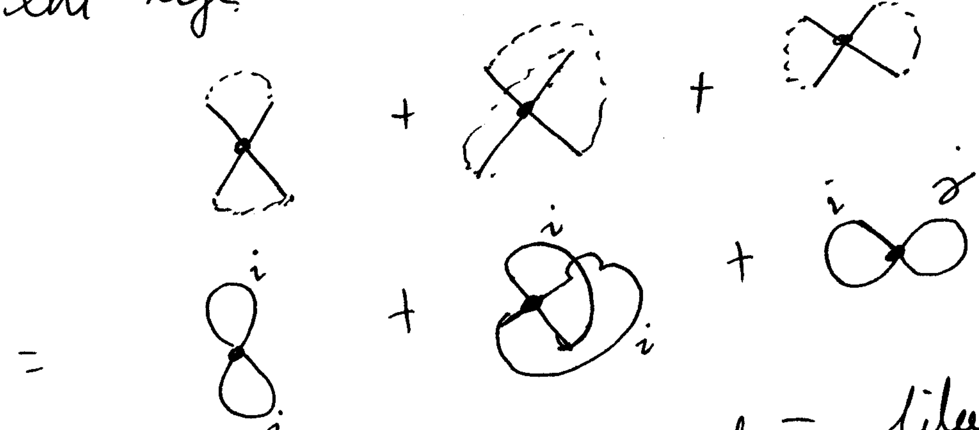
$$+ \langle \phi_i(k_1')\, \phi_i(k_4') \rangle_0 \langle \phi_j(k_2')\, \phi_j(k_3') \rangle_0 .$$

Such terms can be ~~drawn~~ ~~be~~ represented by pictures — the Feynman diagram.

The interaction term is



and the terms can be obtained by connecting the legs



Now each solid line is a term like

$$\langle \phi_i(k_1')\, \phi_j(k_2') \rangle_0$$

This is the "non-interacting" or "bare" propagator

$$\langle \phi_i(k_1') \, \phi_j(k_2') \rangle = G^0(k_1') \, \delta_{ij} \, \delta_{k_1'+k_2', 0}$$

$$\cancel{G} \quad G(k_1') = \frac{1}{(k_1')^2 + t}$$

With this we see that

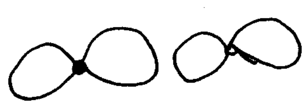
$$\langle \mathcal{H}_u \rangle_0 = \frac{u}{4V} \left[2n \left(\sum_k G^0(k) \right)^2 + n^2 \left(\sum_k G^0(k) \right)^2 \right]$$

↳ factor depending on number of components)

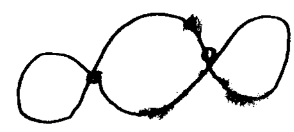
degeneracy of the diagram

To find any term of a particular in the ~~diagram~~ series, draw all unique digrams and obtain their degeneracy.

Note that at any order there will be connected and disconnected digrams. For example, at $m=2$:



disconnected



Connected

The remarkably beautiful thing is that only the connected diagrams ~~(get into the physics)~~ are important.

This can be seen by noting that

$$\ln\left[\frac{Z}{Z_0}\right] = \ln\left[1 - \langle H_u\rangle_0 + \frac{1}{2!}\langle H_u^2\rangle + \cdots\right]$$

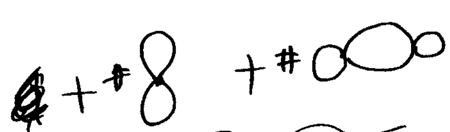
$$= - \langle H_u\rangle_0 + \frac{1}{2!}\langle H_u^2\rangle - \frac{1}{2}\langle H_u\rangle_0^2$$

$$- \langle H_u\rangle_0 + \frac{1}{2}\left(\langle H_u^2\rangle_0 - \langle H_u\rangle_0^2\right)$$

$$+ \cdots$$

Note that $\langle H_u^2\rangle_0 - \langle H_u\rangle_0^2$ will contain only the connected digram.

Thus

$\ln\left[\frac{Z}{Z_0}\right] =$  $+ \cdots$

only connected digram.

This is the Linked Cluster theorem which we have seen before (a slightly different avatar!)

We can use ~~to calculate~~ the same ideas to calculate the Green's function.

$\mathcal{G}_{ij}(k) = \langle \phi_i(k)\, \phi_j(-k) \rangle_{\mathcal{H}}$

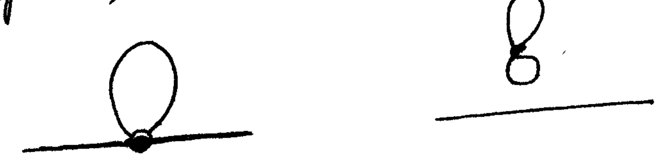
$\uparrow$ ~~all~~ including u

That is

$$\mathcal{G}_{ij}(k) = \frac{\langle \phi_i(k)\, \phi_j(-\vec{k})\, e^{-\cancel{\beta}\mathcal{H}_u} \rangle_0}{Z/Z_0}.$$

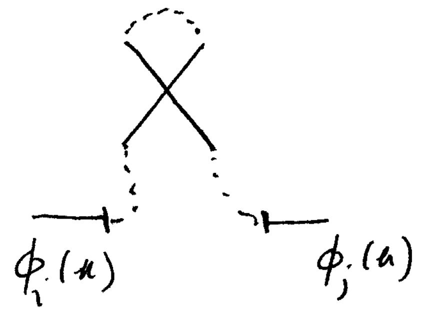
Note that in this case both the numerator aswell as the denominator has to be expanded in powers of u. The key point is that the diagrams in the numerator come in connected and disconnected avatars. The quite remarkable thing, the linked cluster theorem, is that the disconnected diagrams of the numerator are exactly cancelled by the denominator.

For example,



connected disconnected

Can be obtained by connecting

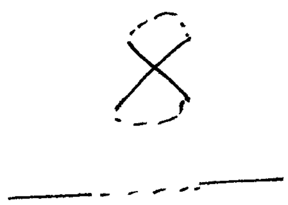


$\longrightarrow$

$$= -\frac{u}{4} G_{ii}(k)\, G(k) \sum_{k_1}{}' G(k_1)$$

Connected

and

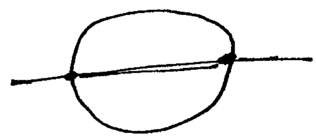


$$= -\frac{u}{4} G(k) \left(\sum_{k_1}{}' G(k) \right)^L$$

Disconnected.

Now the connected diagrams themselves come as one particle reducible or one particle irreducible. For example

1P-reducible

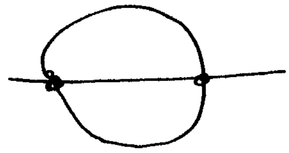


1P irreducible.

The ~~one~~ 1P. reducible digram can be ~~made up of~~ broken into two by cutting a single ~~green's~~ Green's function line.

Now any 1 P irreducible term looks like $G(k)\, F(k)\, G(k)$. For example

$= G(k)$  $G(k)$



$= G(k)$  $G(k)$

The sum of such ~~self energy~~ terms is called the self energy $\Sigma(k)$

The Green's function can now be
expanded as

$$\mathcal{G}_{ij}(k) = \delta_{ij}\,\big(G(k) + G(k)\,\Sigma(k)\,G(k) + G(k)\,\Sigma(k)\,G(k)\,\Sigma(k)\,G(k) + \dots \big)$$

$$\Rightarrow \quad \mathcal{G}^{-1}(k) = G^{-1}(k) - \Sigma(k)$$

This is the famous Dyson equation.
This is all ~~the~~ of the diagrammatic
perturbation theory that we will use.
Let us now see ~~the~~ the nature of
the difficulty in the construction of R_s.
We pose a physics question: In the
absence of u ($u=0$), the t_c is given
by the relation $t=0$. What happens
in the presence of u? We, naturally,
expect t_c to be negative & since u
induces fluctuations. Let us see
a calculation of this quantity. Note
that t_c is given by that value of t
at which the susceptibility & $\chi(k=0)$
diverges. Since we are working in the

symmetric phase, we see that

$$\chi(k) = \langle \phi_i(k)\, \phi_i(-k) \rangle \quad (\text{no } i\text{-sum})$$

But this is just the Green's function

Thus the divergent susceptibility condition can be imposed as follows

Calculate $\chi^{-1}(k)$ and put $\chi^{-1}(k=0) = 0$.

But $\chi^{-1}(k) = \mathcal{G}^{-1}(k) = G^{-1}(k) - \Sigma(k)$

So the problem reduces to the calculation of the self energy..

Since we are near P_G^*, u is small and we can obtain this using perturbation theory keeping only the leading (1st order) in u.

$$\Sigma(k) = $$  $$= -\frac{u}{4V} 2(n+2) \sum_{k'} G(k')$$

$$= -\frac{u}{V} 2(n+2) \frac{V}{(2\pi)^d} \int d^dk\, G(k)$$

$$= -u(n+2) \underbrace{\frac{S_d}{(2\pi)^d}}_{C_d} \int_0^\Lambda dk \frac{k^{d-1}}{k^2+t}.$$

$$\Rightarrow \chi^{-1}(k) = k^2 + t + \underbrace{u(n+2)\, C_d \int_0^\Lambda dk \frac{k^{d-1}}{k^2+t}}_{f(\Lambda, t)}$$

~~Suddenly we realize~~. So, to obtain t_c, we need $\chi^{-1}(k=0)=0$. We get

$$\chi^{-1}(k=0) = t + u(n+2)\, c_d\, f(t,\Lambda)$$

Thus $t_c + u(n+2)\, c_d\, f(\Lambda, t_c) = 0$.

We need to inspect $f(\Lambda, t)$ to solve this equation

$$f(\Lambda,t) = \int_0^{\Lambda} dk \, \frac{k^{d-1}}{k^2+t} \qquad \sim \frac{k^{d-1}}{t} \quad k \simeq 0, \qquad \sim k^{d-3} \quad k \to \infty$$

This is real bad news! $f(\Lambda, t)$ has an <u>ultra-violet</u> divergence for $d-3 \geq -1$ or $d \geq 2$!! It seems that we cannot do perturbation theory in <u>any</u> interesting dimension. The key point is that this is an ultraviolet divergence and arises from physics that we do not know about, and more importantly, we do not care! What do we do? Well we are not the first to encounter this problem! It is one of the key problems faced by early quantum field theory.. Quite ~~too~~ remarkable, this problem can be surmounted by the idea of regularization. We do not treat t as a physical quantity but a "bare" parameter (which may have an infinite constant hiding inside). ~~We get~~ We play the following trick.

Call $r = \chi^{-1}(k=0)$. We have

$$r = t + u(n+2)C_d f(1,t).$$

Invert this relation for r to get

$$t = r - u(n+2)C_d f(1,r).$$

This result is correct to order $O(u^2)$.

Now t_c to order u^2 satisfies the relation

$$t_c = -u(n+2)C_d f(1,0).$$

Thus

$$t - t_c = r - u(n+2)C_d\left[f(1,r) - f(1,0)\right]$$

$$t^{\text{physical}} = r + u(n+2)C_d\left[\int_0^\Lambda dk\, k^{d-1}\frac{r}{k^2(k^2+t)}\right]$$

$$t^{\text{physical}} = r\left[1 + \underbrace{u(n+2)\,C_d \int_0^\Lambda dk \frac{k^{d-1}}{k^2(k^2+r)}}_{F_1(\Lambda, r)}\right]$$

We want the integral $F_1(\Lambda, r)$ to be a nice bounded quantity as $r \to 0$, thus $t^{\text{physical}} \to 0$ as $r \to 0$.

$$\int_0^\Lambda dk \frac{k^{d-1}}{k^2(k^2+r)} = \begin{cases} \dfrac{k^{d-3}}{r} & k \to 0 \\ k^{d-5} & k \to \infty. \end{cases}$$

$F_1(\Lambda, t)$ is well defined in the ultraviolet for $d < 4$, has a $\underline{\log \Lambda}$ divergence in $d = 4$. We say that the theory can be renormalized for $d \leq 4$. Let us stick to $d < 4$. Now we face a diffrent problem! the integral is <u>infrared divergent</u> as $\underline{r \to 0}$! In other words

$$F_1(\Lambda, r) = (n+2)\,C_d\, u \frac{r^{+d/2}}{r^2} \underbrace{\int_0^\infty dx \frac{x^{d-1}}{x^2(1+x^2)}}_{\#}$$

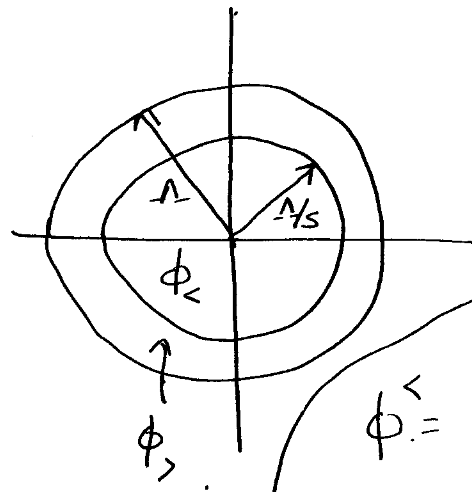
Thus

$$t^{\text{phys}} = r\left[1 + \#(n+2)\,C_d \underbrace{\left[\frac{u}{r^{\frac{4-d}{2}}}\right]}_{u_{\text{eff}}}\right]$$

We see that the effective perturbation parameter $u_{eff} = \frac{u}{r^{\frac{4-d}{2}}}$ diverges as $r \to 0$. In other words perturbation theory is rendered unreliable. Note that this is ~~exact~~ simply the Ginsburg criterion in disguise. The sad thing is that perturbation theory (our favourite tool), ~~mar~~ is not expected to give good results in the critical region. We are now faced with terrible difficulties! When we "cured" the ultraviolet problem, the "infrared" cries out for attention!

The remarkable thing about Wilson's ideas is that, even this infrared problem is addressed - His method provides a way to handle this. Let us therefore plunge into Wilson's calculation. As noted we have to do a three step process to obtain R_s. To do this we introduce some notation that will be used throughout.

We introduce $s > 1$
(s will be close to 1)
and introduce the fields $\phi^<$ and $\phi^>$.



$\phi(k)$
$\phi^< = \quad k < \frac{\Lambda}{s}$
$\phi^> = \phi(k) \quad \frac{\Lambda}{s} < k < \Lambda$

Now

$$\mathcal{H} = H_0 + \mathcal{H}_u.$$

$\downarrow$ symbolically

$$(r+t)\,\phi\phi + u\,\phi\phi\phi\phi.$$

$$\mathcal{H} = \overbrace{(k^2+t)\,\phi^<\phi^< + u\,\phi^<\phi^<\phi^<\phi^<}^{\mathcal{H}^<}$$
$$\underbrace{(k^2+t)\,\phi^>\phi^>}_{\mathcal{H}_0^>} + \overbrace{u\,\phi^>\phi^>\phi^>\phi^>}^{\mathcal{H}_u^>}$$

$$\mathcal{H}_u^{><}\; \left\{ \begin{array}{l} + u\,\phi^<\phi^<\phi^<\phi^> \text{ (and permutations)} \\ + u\,\phi^<\phi^<\phi^>\phi^> \text{ (and permutations)} \\ + u\,\phi^<\phi^>\phi^>\phi^> \text{ (and permutations)} \end{array} \right.$$

$$\mathcal{H} = \mathcal{H}^< + \mathcal{H}_0^> + \underbrace{\mathcal{H}_u^> + \mathcal{H}_u^{><}}_{V}$$

This notation will be used; Note that the term V has in it the fields $\phi^<$.
Let us go through the three steps.

① Integrate.

The key technical idea in this step is the linked cluster theorem.

$$e^{-\mathcal{H}^{<}_{eff}} = \int \mathcal{D}[\phi^{>}]\, e^{-(\mathcal{H}^{<} + \mathcal{H}_0^{>} + V)}$$

$$= e^{-\mathcal{H}^{<}} \int \mathcal{D}[\phi^{>}]\, e^{-[\mathcal{H}_0^{>} + V]}$$

$$\Rightarrow \mathcal{H}^{<}_{eff} = \mathcal{H}^{<} - \ln\left[\int \mathcal{D}[\phi^{>}]\, e^{-[\mathcal{H}_0^{>} + V]}\right]$$

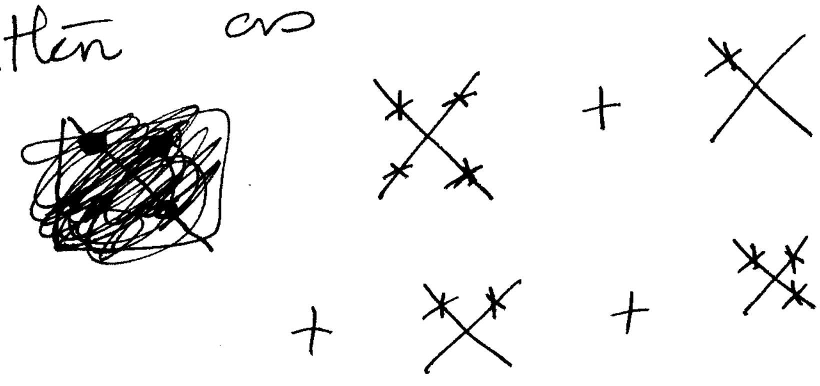
$$\mathcal{H}^{<}_{eff} = \mathcal{H}^{<} - \ln\left[Z_0^{>}\left(\sum_m \frac{(-)^m}{m!}\langle V^m\rangle_0^{>}\right)\right]$$

$$= \mathcal{H}^{<} + \underbrace{\ln Z_0^{>}}_{\text{drop}} + \Big[- \langle V\rangle_0^{>} + \tfrac{1}{2}\left(\langle V^2\rangle_0^{>} - (\langle V\rangle_0^{>})^2\right) + \dots \Big]$$

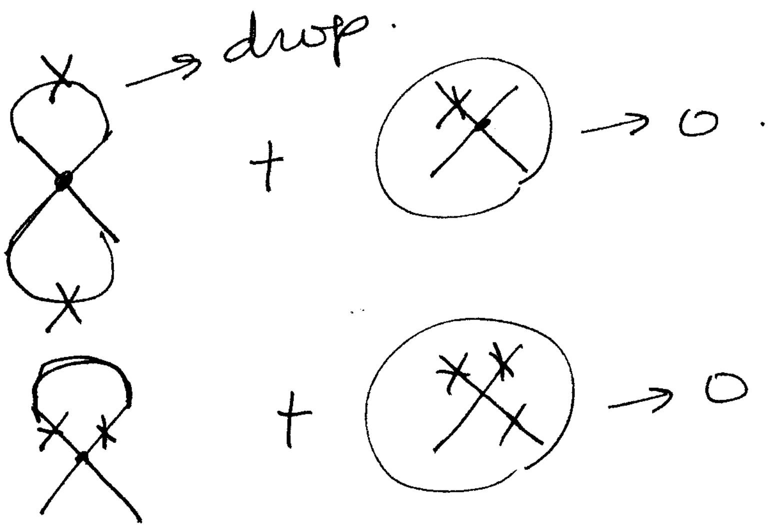
$$\mathcal{H}^{<}_{eff} = \mathcal{H}^{<} + \underbrace{\langle V\rangle_0^{>}}_{O(u)} - \underbrace{\tfrac{1}{2}\left[\langle V^2\rangle_0^{>} + (\langle V\rangle_0^{>})^2\right]}_{O(u^2)}$$

We see that this is the perturbation theory in u, but with only the high k modes i.e. $\Lambda/s < k < \Lambda$

To carry out the calculation we introduce a further diagrammatic notation a line $\ast$ will denote the $\phi^>$ modes and — will correspond to $\phi^<$ modes.. V can then be written as

$V =$ 

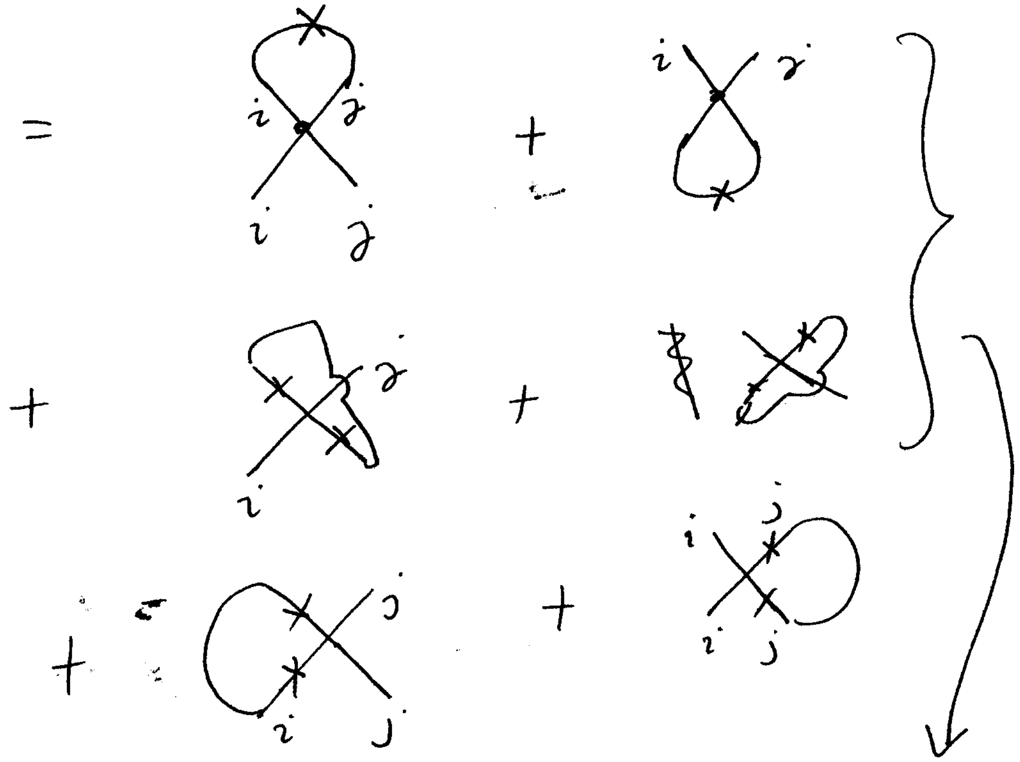
Now

$\langle V \rangle_0^> =$ 

Let us stop at $O(u)$ and "see".

We see that the only term that arises is the term . Let us calculate this term. This is a symbol for a whole bunch of things

$$\langle V \rangle_0^{>} =$$



$$= \frac{u}{4V}\left[4\ \phi_i^<(k_1)\,\phi_i^<(k_2)\left(\sum_{\Lambda/s<k<\Lambda}' G(k)\right)\delta_{k_1+k_2,0}\right]$$

$$+ \frac{u}{4V}\left[2n\ \phi_i^<(k_1)\,\phi_i^<(k_2)\left(\sum_{\Lambda/s<k<\Lambda}' G(k)\right)\delta_{k_1+k_2,0}\right]$$

$$= \phi_i^<(k_1)\,\phi_i^<(k_2)\,\underbrace{\left[\frac{u}{4V}\,2(n+2)\sum_{\Lambda/s<k<\Lambda}' G(k)\right]}$$

~~To order u, $\mathcal{H}_{eff}$~~ $$\frac{u}{2}(n+2)\,C_d\underbrace{\int_{\Lambda/s}^{\Lambda} dk\,\frac{k^{d-1}}{k^2+t}}_{g(\Lambda,t,s)}.$$

The effective $\mathcal{H}_{eff}^<$ hamiltonian is

$$\mathcal{H}^<_{eff} = \sum_{k<\Lambda/s}' (k^2 + t^{eff}) |\phi^<(k)|^2 + \underbrace{\mathcal{H}_u^<}_{\text{unaffected}}$$

$$t^{eff} = t + \frac{u}{2}(n+2)\, c_d\, g(1,t,s)$$

(2) Rescale. We now rescale k-space (and space). $k' = sk$

$$\mathcal{H}' = \sum_{k'}' \frac{1}{2}\left(\left(\frac{k'}{s}\right)^2 + t^{eff}\right) |\phi^<(k)|^2 + \mathcal{H}_u^<.$$

(3) Renormalise

We set the coefficient of $(k')^2$ to be unity. $\phi^<(k) = \zeta(s)\, \phi'(k')$

$\zeta(s) = s$

$$\mathcal{H}' = \sum_{k'}' \frac{1}{2}\left((k')^2 + \overbrace{s^2 t^{eff}}^{t'}\right) |\phi'(k')|^2 + \frac{(s^{(4-d)} u)}{V'} \sum_{k'}' \phi_i(k_1)\phi_j(k_2)\phi_j(k_3)\phi_i(k_4)\, \delta_{k_1+k_2+k_3+k_4,0}.$$

We have thus obtained $\mathcal{H}'$ to have the same form as $\mathcal{H}$

We see that

$$R_s[\{t, u\}] = \{s^2(t + u(n+2)C_d\, g(\Lambda, t, s)),\ s^{(4-d)} u\}.$$

To find the flow equations we set

$$s = e^{l}$$

$$\frac{dt}{dl} = s\frac{dt}{ds} = 2t + (n+2)C_d\left(\frac{dg}{ds}\right)\Big|_1 u$$

$$\frac{du}{dl} = (4-d)u.$$

$$\left.\frac{dg}{ds}\right|_{s=1} = \cancel{s\frac{d}{ds}} - \frac{\Lambda^{d-1}}{\Lambda^2/s^2 + t}\left(-\frac{\Lambda}{s^2}\right)\Big|_{\text{lims}\to 1}$$

$$= \frac{\Lambda^d}{\Lambda^2 + t} \to \Lambda^{d-2}.$$

Thus

$$\left.\begin{aligned} \frac{dt}{dl} &= 2t + C_d(n+2)\, u\Lambda^{d-2} \\ \frac{du}{dl} &= (4-d)u \end{aligned}\right] \text{ to } O(u^2).$$

Fixed points

$$2t + C_d (n+2)\Lambda^{d-2} u = 0$$

$$(4-d)\, u = 0.$$

$\Rightarrow$ $t^* = 0$ and $u^* = 0$ which is gaussian fixed point p^*_G ! We see ~~however that the~~ both the operators are relevant. The key finding is that the eigen directions are different

$$\begin{pmatrix} \delta t' \\ \delta u' \end{pmatrix} = \begin{pmatrix} 2 & C_d (n+2)\Lambda^{d-2} \\ 0 & (4-d) \end{pmatrix} \begin{pmatrix} \delta t \\ \delta u \end{pmatrix}$$

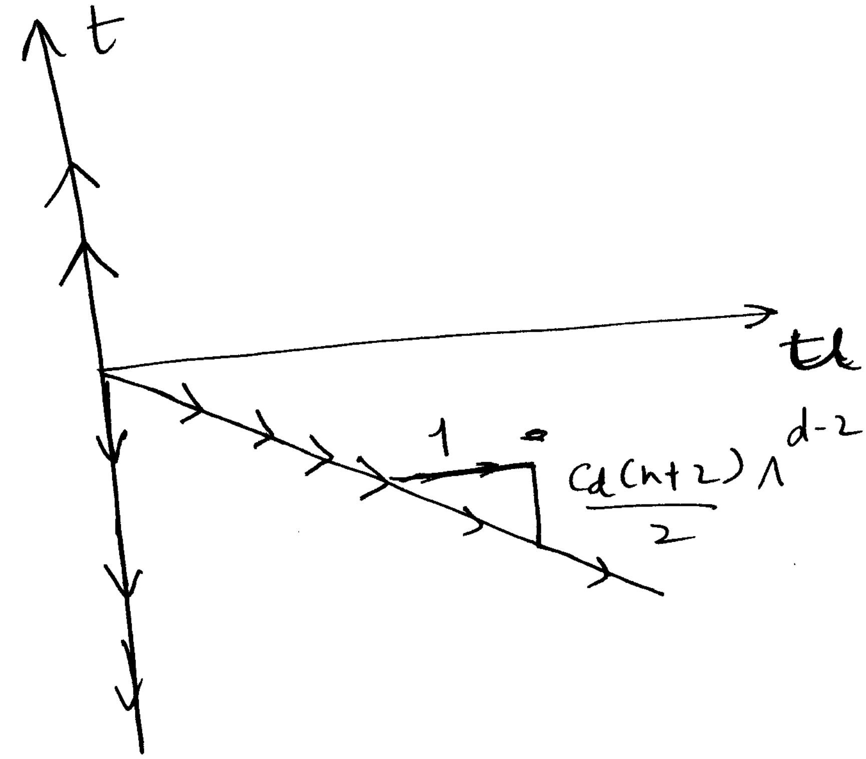
$\mu_t = 2$ and $\mu_u = 4-d.$

The eigen direction corresponding to $\mu_t = 2$ is $\delta u = 0$ and the eigendirection corresponding to $\mu_u = 4-d$ is obtained

$$2\,\delta t + C_d (n+2)\Lambda^{d-2} \delta u = 0.$$

$$\Rightarrow \delta t = -\frac{C_d (n+2)\Lambda^{d-2}}{2} \delta u.$$

We see that the flow digram is



We see that the relevant operator (which is a linear combination of t and u) takes us towards <u>negative</u> values of t. This is <u>exactly</u> what we expected! ~~Eta~~ u induces fluctuations which should reduce t_c from the Gaussian value.

But now we are faced with the question, where does the flow take us? if we are in $d<4$ (not there will be no flow away from P_a^* at this order if $d=4$). We know that it is towards increasing value of u. (this was also anticipated). Now one is really worried if perturbation theory will work. Well, one hopes that the flow will take us to a nontrivial critical point with u^* as the coupling. The key question is if u^* is "small"? One can argue and, we will see later explicitly, that the flow equation for u to quadratic order is

$$\frac{du}{dl} = (4-d)\,u - \underset{\text{some constant}}{A}\,u^2$$

Thus $$u^* = \frac{4-d}{A} = \frac{\varepsilon}{A} \quad (\varepsilon = 4-d)$$

This led Wilson and Fischer to make a bold step forward, namely the use of $\varepsilon = 4-d$ as an expansion parameter. The idea is to treat ε as small and carry out all calculations to some order in ε, and then set the value of ε to its value (4-d).

This now goes under the name "ε-Expansion".

With this in mind let us proceed with the calculation to 2nd order in u.

① ~~Integ~~ Integrate.

The second order contribution to $\mathcal{H}_{eff}^{<}$ are the connected second order diagrams.

$$\langle V^2 \rangle_{>} = \langle (\mathcal{H}_u^{>} + \mathcal{H}_u^{><})^2 \rangle_{>}$$

$$\langle (\mathcal{H}_u^{>})^2 + 2\mathcal{H}_u^{>}\mathcal{H}_u^{><} + (\mathcal{H}_u^{><})^2 \rangle_{>}$$

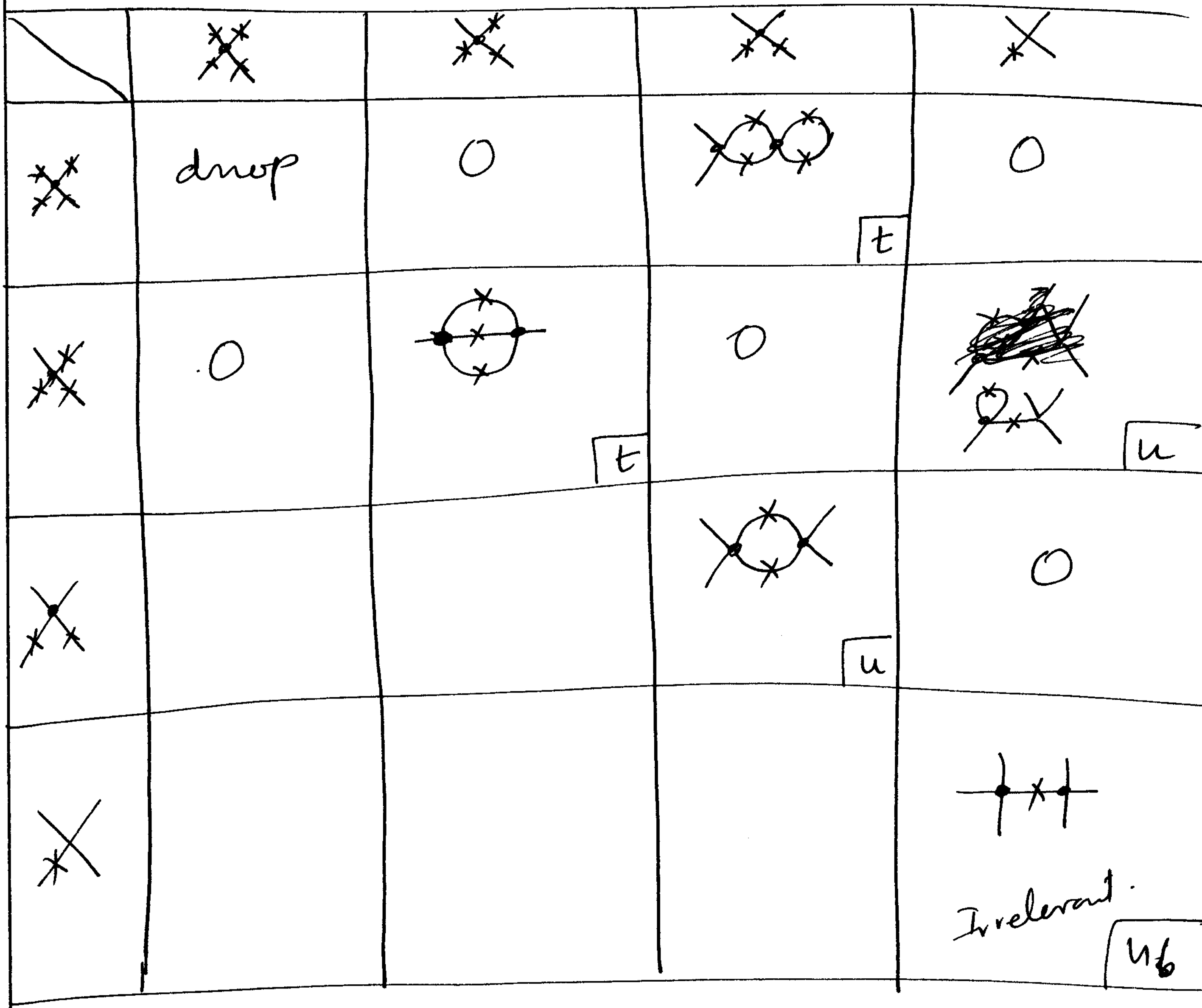
↑ drop gives only constant term

↳ contributes only to t

contributes to t and u

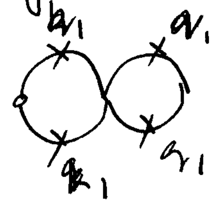
All of these can be treated together by treating them as products of u.

One can draw a ~~picture of this~~ table of this.



(Unfilled boxes are "symmetric")

Diagrams contributing to t

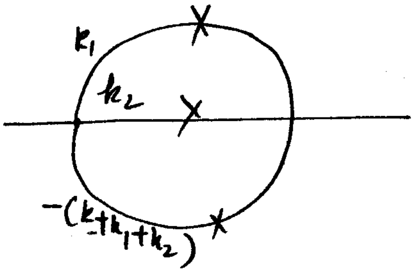


$$= \frac{u^2}{16V^2}\left(\underbrace{\sum_1 \cancel{t}\, G^2(q_1)}_{h(}\right)\sum_1 G(q_1)$$

Factors. $\cancel{2/6}\; 1 F_{1x}(n,d)\; u^2\; h(\Lambda,t,s)\, g(\Lambda,t,s)$

$\sim$ 2 ↑ factors (we don't need them so we will not evaluate)

Now the second term is



$$\sim \frac{u^2}{16V^2} \sum_{\substack{\bigcirc \\ k_1,k_2}}{}' \frac{1}{k_1^2+t}\,\frac{1}{k_2^2+t}\,\frac{1}{(k+k_1+k_2)^2+t}$$

Note that by symmetry the sum is

$$f_2(n,d)\,\frac{u^2}{16} \int_{\bigcirc} d^dk_1\, d^dk_2\, \frac{1}{k_1^2+t}\,\frac{1}{k_2^2+t}\,\frac{1}{(k+k_1+k_2)^2+t}$$

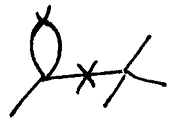
↑
factors

$$\frac{1}{2} f_2(n,d)\,\frac{u^2}{\cancel{16}} \left[A(\Lambda, t) + B(\Lambda, t, s)\, k^2 \right].$$

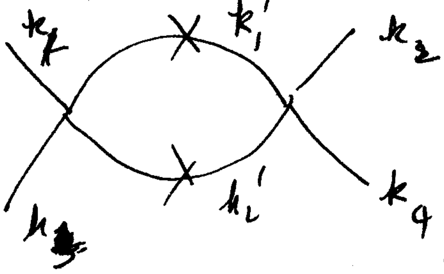
factor

This form is obtained by noting that integral does not change in going from $k \to -k$.

We now see that this changes the coefficient of k^2 and will produce a non trivial field renormalization of $O(u^2)$..

Let us now see what happens to u. It gets contribution from 

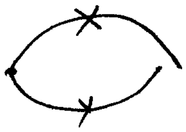
But one can show that this has no small momentum piece and is hence dropped. Finally we get



$\delta_{k_1+k_3+k_1'+k_2',0}$

$\delta_{k_1'+k_2'+k_2+k_4,0}$ this is

For nearly zero incoming momentum



$$= V \frac{u^2}{16V^2(8\pi)} V^{-1} \sum_{k_1', k_2'} \frac{1}{k_1'^2+t}\,\frac{1}{k_2'^2+t}\,\delta_{k_1'+k_2',\,-(k_1+k_3)}$$

$$k_2' = \left(-k_1' - (k_1+k_3)\right)$$

$$= \frac{u^2}{16V} K_d \int d^dk \, \frac{1}{(k^2+t)\left((k+(k_1+k_3))^2+t\right)^2}$$

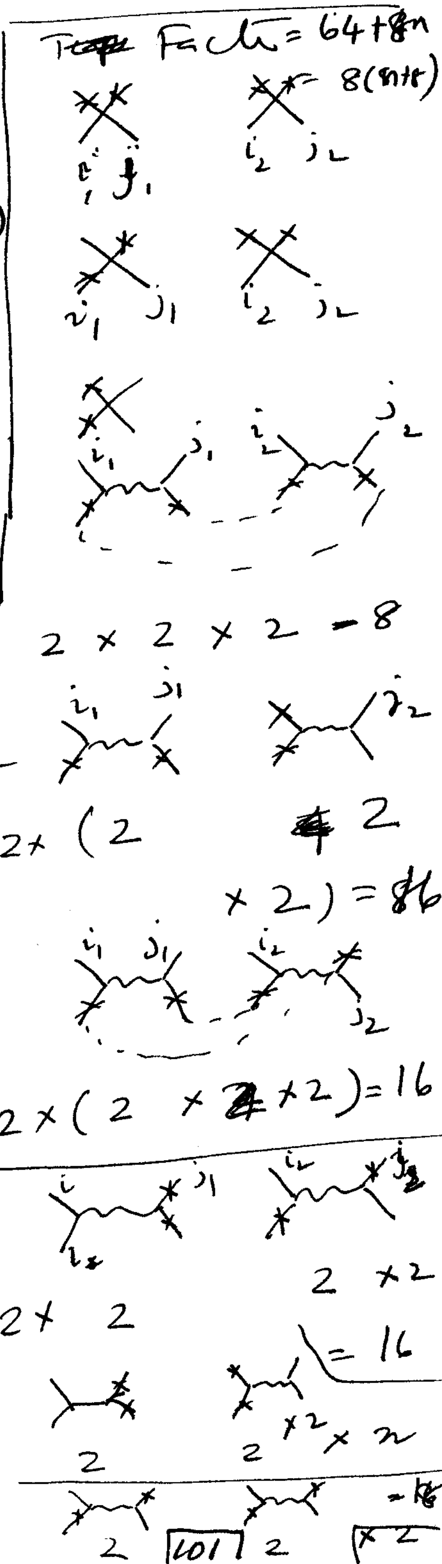
We need only $k_1+k_2 \sim 0$

$$= \frac{u^2}{16V} K_d \int d^dk \, \frac{1}{(k^2+t)^2}$$

↓ fact

$$= \cancel{F_3(n)}\, \frac{u^2}{16V} K_d \underbrace{\int_{\Lambda/s}^{\Lambda} d^dk \, \frac{1}{(k^2+t)^2}}_{h(\Lambda,t,s)}$$

$$F_3(n) = 8(n+8)$$



We thus have

~~$R_s[\{t, u\}] =$~~

$$\begin{cases} t^{eff} = t + u(n+2)c_d\, g(1,t,s) \\ \qquad + F_1(n,d)\, u^2 h(1,t,s) g(1,t,s) \end{cases}$$

$$u^{eff} = u - (n+8)\, c_d\, u^2 h(1,t,s).$$

After rescaling and renormalisation (we ~~dont~~ do ~~not~~ not do field renormalisation) (see later why)

we get

$$\frac{dt}{dl} = 2t + c_d (n+2) \frac{\Lambda^{d} u}{\Lambda^2 + t} + O(u^2)$$

$$\frac{du}{dl} = (4-d)\, u - (n+8)\, c_d\, u^2 \frac{\Lambda^{d}}{(\Lambda^2+t)^2} \left. \frac{d}{ds} h(1,t,s) \right|_{s=1}$$

~~We see that $u^* = 0$ or $u^* = \frac{(4-d)\Lambda^{4-d}}{(n+8)c_d}$~~

~~and $t^* = \frac{\ldots(4-d)\Lambda^{4-d}}{}$~~

$$u^* = \frac{\varepsilon \Lambda^{\varepsilon}}{(n+8) C_4 \varepsilon} \simeq \frac{\varepsilon\, (e^{\varepsilon \ln \Lambda})}{(n+8)\, C_4}$$

$$\simeq \frac{\varepsilon}{(n+8)\, C_4} \quad \text{to order } \varepsilon.$$

$$t^* = -\frac{(n+2)}{2(n+8)} \Lambda^2 \cdot \varepsilon \quad \text{to order } \varepsilon.$$

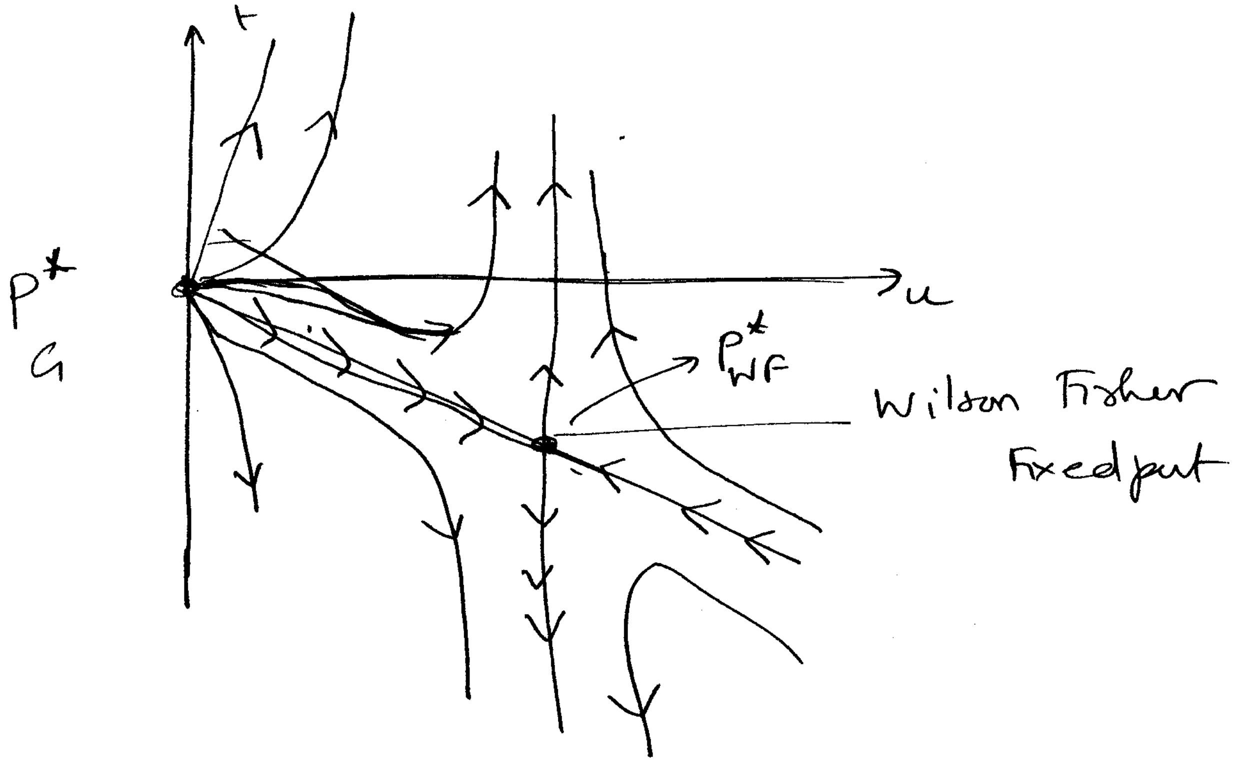
We thus see that

$$\frac{d}{dl}\begin{pmatrix} \delta u \\ \delta t \end{pmatrix} = \left[\begin{array}{c|c} 2 + \frac{C_d (n+2)\varepsilon}{C_d (n+8)} & \frac{C_d (n+2)}{\Lambda^{d-2}} \\ \hline O(\varepsilon^2) & -\varepsilon \end{array}\right] \begin{bmatrix} \delta u \\ \delta t \end{bmatrix}.$$

We get $\mu_t = 2 - \frac{(n+2)\varepsilon}{(n+8)} + O(\varepsilon^2)$

and $\mu_u = -\varepsilon.$

We see that t is relevant and u is irrelevant. The full flow diagram is



Note that $u^*_{WF} \sim \varepsilon$

We can now calculate the critical exponent:

The correlation length exponent is

$$\nu = \frac{1}{\mu_t} = \frac{1}{2 - \frac{(n+2)}{n+8}\varepsilon}$$

$$\simeq \frac{1}{2}\left[1 + \frac{(n+2)}{2(n+8)}\varepsilon\right] + O(\varepsilon^2)$$

$$\alpha = 2 - \frac{d}{\mu_t}$$

$$= 2 - \frac{(4-\varepsilon)}{\mu_t} = 2 - 2 - \frac{(n+2)\varepsilon}{(n+8)} + \frac{1}{2}\varepsilon$$

$$= \varepsilon\left[\frac{4-n}{2(n+8)}\right] + O(\varepsilon^2)$$

We can easily calculate other exponent

We find that

$$\frac{dh}{dl} = +\underbrace{\left(\frac{d}{2}+1\right)}\, h$$

$$\mu_h = \left(\frac{d}{2}+1\right) = \frac{4-\varepsilon}{2}+1 = \left(3-\frac{\varepsilon}{2}\right)\cdots$$

$$\beta = \frac{d-\mu_h}{\mu_t} = \frac{1}{2} - \frac{3}{2(n+8)}\varepsilon$$

$$\gamma = \frac{2\mu_h - d}{\mu_t} \simeq 1 + \frac{(n+2)}{2(n+8)}\varepsilon$$

$$\delta = \frac{1}{\mu_h} - 1 = \frac{\mu_h}{\mu_h - 1} = \frac{\left(3-\frac{\varepsilon}{2}\right)}{2-\frac{\varepsilon}{2}}$$

$$= \left(\frac{3}{2} - \frac{\varepsilon}{4}\right)\left(1+\frac{\varepsilon}{2}\right)$$

$$= \frac{3}{2} + \frac{\varepsilon}{4}.$$

$$\eta = O(\varepsilon^2) = 0$$ at this order.

For ising model in $d=3$, $(n=1)$,

we get

$$\alpha = \cancel{\frac{1}{2}\left[1+\frac{1}{6}\right]} \qquad \frac{3}{18} = \frac{1}{6} \simeq 0.17$$

This close to the known value of 0.11.

We thus see that ~~the~~ RG provides a nice framework to understand universality (most operators are irrelevant) and also obtain numerical values of critical exponents. Some comment are in order

① We calculate the flow to order u^2 and found that this involved doing two k-sums. A k-sum is called a "loop". What we did is a 2-loop calculation to $O(\varepsilon)$.

② Going to higer loops becomes cumbersome as the constraint of being in the ◯ becomes harder to impose. Here we go to the more field theoretic techniques and can go on to higer orders. The record is held ~~b~~ by Kleinert who went on to 6 loops!

(3) There are also other ways of controlling the perturbative expansion. We note the $u^* = \frac{\varepsilon}{n+8}$ and hence we can also use $1/n$ expansion where n is the number of components.

~~(4) Note that ...~~

(4) We have not calculated η, or better said to the order of our calculation $O(\varepsilon)$, $\eta = 0$. The field theoretic method gives

$$\eta = \frac{(n+2)}{2(n+8)^2}\,\varepsilon^2$$

Even for the ising model

$$\eta = \frac{1}{54}$$

and is quite small. (numerical value $\eta = 0.06$)

It is ~~quite~~ interesting to note that although the field renormalization, ~~occurs~~ and anomalous dimension occurs at $O(\varepsilon^2)$, other qualities such as t, pick ~~up~~ have anomalous dimension