Fluids & Plasmas, AA 363

Problem Set I: Fluid Kinematics, Vorticity, Equilibrium

1. **Streamlines:** Let the streamlines of a time-independent 2-D flow on $\mathbb{R}^2$ be given by $(u, v) = (\alpha x, -\alpha y)$. Sketch the streamlines. Express, as a function of time, the trajectory of a fluid element starting at $(1, 1)$ at $t = 0$. How does the density of this fluid element vary as a function of time? How do the sides of a rectangular fluid element vary as it moves along streamlines?

2. **Volume and Density:** Consider a fluid element with density ρ_0 centered at (x_0, y_0, z_0). Assume that the fluid element flows to (x, y, z) at a later time. Show that the density ρ at the later time is related to ρ_0 as
$$\rho = \rho_0 \frac{\partial(x_0, y_0, z_0)}{\partial(x, y, z)},$$
where the right-hand-side denotes the Jacobian connecting the two volume elements (adapted from Batchelor).

3. **Euler Equations in Conservative Form:** Starting with the standard form of Euler equations,
$$\frac{d\rho}{dt} = -\rho(\vec{\nabla} \cdot \vec{u}), \quad \frac{d\vec{u}}{dt} = -\vec{\nabla} p/\rho, \quad \rho \frac{d\epsilon}{dt} = -p(\vec{\nabla} \cdot \vec{u}),$$
derive the conservative form of Euler equations,
$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{u}) = 0, \quad \frac{\partial \rho \vec{u}}{\partial t} + \vec{\nabla} \cdot \left(\rho \vec{u} \vec{u} + p \overleftrightarrow{I} \right) = 0, \quad \frac{\partial}{\partial t} \left(\rho \epsilon + \frac{1}{2} \rho u^2 \right) + \vec{\nabla} \cdot \left[\rho \vec{u} (u^2/2 + w) \right] = 0,$$
where the sumbols have their usual meaning. Why is this the conservative form?

4. **Self-gravitating, Rotating Disk:** A useful model for the warm interstellar medium (ISM) is that it is a rotating, self-gravitating, equilibrium flow in form of a thin disk. Under the thin disk approximation the vertical and radial directions are decoupled. The vertical force and Poisson's equations are given by
$$\frac{dp}{dz} = -\rho \frac{d\phi}{dz}, \quad \frac{d^2 \phi}{dz^2} = 4\pi G \rho.$$
Assuming WIM to be isothermal (constant temperature T) show that the density have a $\rho = \rho_0 \mathrm{sech}^2(z/z_0)$ profile in the $z-$ direction. Express z_0 in terms of T, ρ_0 and G.

5. **Potential Flows in 2D:** For incompressible and irrotational flows, $\vec{\nabla} \cdot \vec{u} = \vec{\nabla} \times \vec{u} = 0$. This implies that $\vec{u}$ can be written as $-\vec{\nabla}\phi$ (where ϕ is called the velocity potential) and the incompressibility condition implies $\nabla^2 \phi = 0$; i.e., ϕ satisfies the Laplace's equation. From your studies of electrostatics, recall that this equation can be solved if we specify the value of ϕ (Dirichlet BC) of $\partial\phi/\partial n$ (Neumann BC) at the boundaries. Now, solve for the steady streamlines for a flow past a spherical surface of radius a. The normal velocity at the surface of the sphere should vanish.

6. **Rotating Bucket:** An ideal fluid is rotating under gravity g with constant angular velocity Ω, so that relative to fixed Cartesian axes $\vec{u} = (\Omega y, \Omega x, 0)$. We wish to find the surfaces of constant pressure, and hence the surface of a uniformly rotating bucket of water (which will be at the atmospheric pressure).

   'By Bernoulli,' $p/\rho + u^2/2 + gz$ is a constant, so the constant pressure surfaces are
$$z = \text{constant} - \frac{\Omega^2}{2g}(x^2 + y^2).$$

But this means that the surface of a rotating bucket of water is at its highest in the middle. What is wrong?

Write down the Euler equations in component form, integrate them directly to find p and hence find the correct shape of the free surface. (Problem 1.2 in Acheson)