Fluids & Plasmas, AA 363

Problem Set III: Viscous Flows, Gasdynamics

1. **Taylor-Couette flow:** Consider a viscous fluid between two infinite co-axial cylinders with radii a and b. If the two cylinders are rotated with Ω_a and Ω_b respectively, determine the velocity distribution between the cylinders. You will need to write the Navier-Stokes equation in cylindrical coordinates. Now assume that the outer cylinder is non-rotating and the inner cylinder suddenly started into rotation with angular velocity Ω from a state of rest. Describe how the vorticity changes in time. (adapted from Arnab's book & Acheson).

2. **Stokes drag & terminal speed:** A rigid ball of radius a falls inside an infinite viscous fluid in a constant gravitational field. Using the form of Stokes viscous drag (using dimensional analysis) write down the equation of motion and show that the ball attains a terminal velocity $\propto a^2$; i.e., a smaller ball falls more slowly (adapted from Arnab's book).

3. **Viscous flow layers:** Two incompressible viscous flows of the same density ρ flow, one on top of the other, down an inclined plane making an angle α with the horizontal. Their viscosities are μ_1 and μ_2, the lower fluid is of depth h_1 and the upper fluid is of depth h_2. Show that

$$u_1(y) = [(h_1 + h_2)y - y^2/2]\frac{g \sin \alpha}{\nu_1},$$

so that the velocity of the lower fluid $u_1(y)$ is dependent on the depth h_2, but not the viscosity, of the upper fluid. Why is this?

4. **Energy dissipation:** Consider an incompressible fluid in the absence of a body force; i.e., $\vec{F} = 0$. Starting from the Navier-Stokes equation with the viscous term show that the rate of change of kinetic energy density of the fluid is

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho u^2\right) = -\frac{\partial}{\partial x_j}\left[\rho u_j \left(\frac{u^2}{2} + \frac{p}{\rho}\right) + u_j \Pi_{ij}\right] + \Pi_{ij}\frac{\partial u_i}{\partial x_j}.$$

Argue from this that the rate of energy dissipation per unit volume is given by

$$-\Pi_{ij}\frac{\partial u_i}{\partial x_j} = \frac{\mu}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right),$$

on using the definition of Π_{ij}. This term represents that fact that the kinetic energy is dissipated due to viscosity (friction between layers), and the internal energy increases at its expense. This is very similar to the problem in HW1 where internal energy density equation was derived from Boltzmann equation (adapted from Arnab's book).

5. **Shock jump conditions:** Use the Rankine-Hugoniot relations to show that the downstream Mach number $\mathcal{M}_2$ at a shock obeys the following:

$$\mathcal{M}_2^2 = \frac{(\gamma - 1)\mathcal{M}_1^2 + 2}{2\gamma\mathcal{M}_1^2 - (\gamma - 1)}$$

and show that this relation can be written as $X_1 X_2 = 1$, where

$$X = \frac{2\gamma}{\gamma + 1}(\mathcal{M}^2 - 1) + 1.$$

Express ρ_2/ρ_1 and p_2/p_1 in terms of X_1. What is the allowable range of X_1? Show that the entropy change through the shock is given by

$$\frac{1}{c_v}(S_2 - S_1) = \ln X_1 - \gamma \ln \left[\frac{(\gamma+1)X_1 + (\gamma-1)}{(\gamma-1)X_1 + (\gamma+1)} \right]$$

and deduce that only compressive shocks ($\rho_2 > \rho_1$) occur in nature (from Pringle & King).

6. **Spherical/Bondi accretion:** A fluid flow with radial velocity $u(r)$ and density $\rho(r)$ represents the steady, spherically symmetric (Bondi) accretion of isentropic fluid from a surrounding medium of uniform density ρ_∞ onto a gravitating point mass M centered at the origin ($r = 0$). Show that $u(r)$ and the adiabatic sound speed $c_s(r)$ obey the following equation:

$$\frac{1}{u}\frac{du}{dr} = \frac{1}{r} \cdot \frac{GM/r - 2c_s^2}{c_s^2 - u^2},$$

and find the corresponding equation for dc_s/dr. Show that at the radius r at which the flow is trans-sonic, the velocity is given by $u^2 = GM/2r$. Verify that at large radii the equations permit a solution of the form $u \to 0$ and $\rho \to \rho_\infty$ as $r \to \infty$. Verify that at small radii the equations permit a solution of the form $u^2 \sim GM/r$ and $u^2 \gg c_s^2$ as $r \to 0$, provided that the ratio of specific heats, γ, is such that $\gamma < \gamma_{\rm crit}$, where the value of $\gamma_{\rm crit}$ is to be determined (from Pringle & King).