Fluids & Plasmas, AA 363

Problem Set IV: Waves and Instabilities

1. **RTI in a stratified atmosphere:** Consider a pressure-supported, hydrostatic gas against gravity. How does the pressure vary as height in equilibrium? Now we want to carry out a linear stability analysis by assuming the perturbations to vary as $\exp(-iwt + ik_x x + ik_y y)$ as the $x-$ and $y-$ directions are homogeneous but not the $z-$ direction. To simplify the calculation we assume $\vec{\nabla} \cdot \vec{u} = 0$. Write down the linearized versions of equations of mass conservation and momentum in the three directions. Set up an eigenvalue problem in the vertical direction in terms of the perturbed density ρ_1. Solve the eigenvalue problem (i.e., obtain the growth rate) using the boundary condition that u_z vanishes at $z = 0$, L and the equilibrium density given by $\rho = \rho_0 e^{x/a}$. For what sign of a is the equilibrium stable? Which is the fastest growing mode for a given $\vec{k}$? How does the growth time compare with the free-fall time? Not that the set up here is different from the two interfaces separated by an interface, that we considered in class.

2. **Group and phase velocity:** Recall the dispersion relation for internal gravity waves that we derived in class
$$w^2 = \frac{k_\perp^2}{k^2} N^2,$$
where N is the Brunt-Vaisala frequency and $\vec{k}_\perp = k_x \hat{x} + k_y \hat{y}$. Calculate the phase velocity and the group velocity for these waves and show that the directions in which the crest/troughs travel is perpendicular to the direction in which the energy is transported.

3. **Particle paths for surface-gravity waves:** Show that the perturbed velocities for surface gravity waves can be written as
$$u = Awe^{ky} \cos(kx - wt); \qquad v = Awe^{ky} \sin(kx - wt); \qquad (y < 0)$$
where symbols have their usual meanings. Assuming that fluid elements depart only a small amount (x_1, y_1) from their equilibrium values, show that the particle paths are circular and that the radius of the circle decreases with depth.

4. **Surface gravity waves with finite depth:** Show that the dispersion relation for the surface waves in a fluid of uniform depth h is
$$w^2 = gk \tanh(kh),$$
where symbols have their usual meanings.

5. **Wave propagation:** Surface waves generated by a mid-Atlantic storm arrive at the British coast with period 15 seconds. A day later the period of the waves arriving has dropped to 12.5 seconds. Roughly how far away did the storm occur (from Acheson)?

6. **Steady corrugated flow:** Water flows steadily with speed U over a corrugated bed $y = -h + \epsilon \cos kx$, where $\epsilon \ll h$, so that there is a time-independent disturbance $\eta(x)$ to the free surface, which would be at $y = 0$ but for the corrugations. By writing
$$u = U + \frac{\partial \phi}{\partial x}, \qquad v = \frac{\partial \phi}{\partial y},$$
where $\phi(x, y)$ is the velocity potential of the disturbance to the uniform flow, show that the linearized boundary conditions are
$$U\frac{d\eta}{dx} = \frac{\partial \phi}{\partial y}, \qquad U\frac{\partial \phi}{\partial x} + g\eta = 0 \;\text{ on } y = 0, \frac{\partial \phi}{\partial y} = -Uk\epsilon \sin kx \;\text{ on } y = -h,$$

and hence find $\eta(x)$. Deduce that crests on the free surface occur immediately above troughs on the bed if

$$U^2 < \frac{g}{k} \tanh(kh),$$

but that if this inequality is reversed the crests on the surface overlie the crests on the bed (from Acheson).

7. **Inertial waves in a rotating flow:** Suppose an inviscid incompressible fluid is rotating uniformly with angular velocity $\vec{\Omega}$, and take Cartesian axes fixed in a frame rotating with that angular velocity. Assume (can you motivate this?) that the small velocity $\vec{u}_1$ relative to the rotating axes satisfies,

$$\frac{\partial \vec{u}_1}{\partial t} + 2\vec{\Omega} \times \vec{u}_1 = -\frac{1}{\rho}\vec{\nabla} p_1,$$

where p_1 is the 'reduced pressure.' Write out these equations in Cartesian components, taking $\vec{\Omega} = (0, 0, \Omega)$, and eliminating u_1, v_1 and w_1 show that

$$\left[\frac{\partial^2}{\partial t^2}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) + 4\Omega^2 \frac{\partial^2}{\partial z^2}\right] p_1 = 0.$$

Hence show that plane waves $\propto e^{i(kx+ly+mx-wt)}$ are possible if

$$w^2 = \frac{4\Omega^2 m^2}{k^2 + l^2 + m^2},$$

and deduce that the group velocity of a packet of such waves is perpendicular to the phase velocity (from Acheson).