Fluids & Plasmas, AA 363

Problem Set VI: Basics of plasmas, orbit theory

1. Find the Debye lengths $\lambda_{D,e}$ and $\lambda_{D,p}$, the plasma parameters Λ_e, Λ_p, the plasma frequencies $\nu_{p,e}$, $\nu_{p,p}$, the thermal velocities $v_{th} = (kT/m)^{1/2}$, the gyrofrequency and gyroradii for electrons and protons in the following conditions (from Sturrock):

   (a) a fusion machine: $n_e = n_p = 10^{16}$ cm^{-3}, $T_e = T_p = 10^7$ K, $B = 10^4$ G.

   (b) the center of the sun: $n_e = n_p = 10^{26}$ cm^{-3}, $T_e = T_p = 10^{7.2}$ K, $B = 10^6$ G.

   (c) the solar wind: $n_e = n_p = 10$ cm^{-3} cm, $T_e = T_p = 10^5$ K, $B = 10^{-5}$ G.

   (d) the atmosphere of a neutron star: $n_e = n_p = 10^{12}$ cm^{-3}, $T_e = T_p = 10^7$ K, $B = 10^{12}$ G.

   Can all these be considered as plasmas? Compare gyrofrequencies with plasma frequencies.

2. Consider the orthogonal uniform electric and magnetic fields $\vec{E} = (0, E, 0)$, $\vec{B} = (0, 0, B)$.

   (a) Assuming $E < B$, find the velocity of the moving frame in which the electric field is zero. What is the (relativistically correct) motion of the particle in this frame? What would be the corresponding motion in the rest/lab frame? Relate this to the E cross B drift.

   (b) Assuming $E > B$, find the velocity of the moving frame in which the magnetic field is zero. What is the motion of the particle in this frame? What would be the corresponding motion in the rest/lab frame? (from Sturrock)

3. Magnetic field is given by $B_z(y) = H(y) - 1/2$ (such a magnetic field appears during reconnection), where H is the Heaviside function. Draw electron and ion orbits starting at $y = 0$ with a non-zero velocity. Is there any relation of the orbits to grad-B drift? Recall that grad-B drift was derived assuming that the variation in B happens over a scale much bigger than the Larmor radius.

4. **Adiabatic invariants & drifts:** Consider a cylindrically symmetric coil that produces a field that is uniform within a certain volume. At time $t = t_0$, the field has a small initial value B_0, and a particle is gyrating in a small circular orbit of radius r_0 at a distance R_0 from the axis of symmetry. The magnetic field slowly increases linearly with time to reach a value B_1 at time t_1: a) calculate $E_\phi(R, t)$; b) calculate the drift velocity due to the combined action of the electric and magnetic fields; c) calculate R_1, the final value of the distance of the particle from the symmetry axis; d) what quantity (in addition to the magnetic moment) is invariant as a result of the drift motion? e) is your analysis relativistically correct? (from Sturrock)

5. **Magnetic pumping:** Consider a distribution of particles that all have the same initial kinetic energy U_0 but have velocity vectors that are distributed isotropically. Consider the following sequence of events: i) the magnetic field strength is increased by the factor λ; ii) the particle distribution is isotropized (e.g., via collisions or scattering by magnetic inhomogeneities); iii) the magnetic field strength is slowly returned to the original value; iv) the distribution is again isotropized.

   (a) Express the final mean energy is U_1 in terms of U_0 and λ.

   (b) Show that $U_1 > U_0$.

   (c) The particles in fact have a distribution of energy (i.e., are no longer mono-energetic). What are the maximum and minimum energies? (from Sturrock)

6. **Mirror confinement:** Consider a mirror machine of length $2L$ with a mirror ratio of 10, so that $B(L) = B(-L) = 10B_0$. A group of $N(\gg 1)$ electrons with an isotropic velocity distribution is released at the center of the machine. Ignoring collisions and space charges, how many electrons escape? Sketch the velocity space distribution function of particles at the center and at the maximum field strength. (from Sturrock)

7. **Larmor radius, confinement & acceleration:** Cosmic rays (relativistic particles from space) are expected to be accelerated in environments where they can be confined till they can be accelerated to the desired energies. What should be the size of the system if we want to accelerate protons to 10^{19} eV with a magnetic field strength of 1 μG and with 10^{13} G. What is the result if we have fully ionized iron nuclei or electrons instead of protons?

8. **Adiabaticity:** A hydrogen plasma is heated by applying a radio frequency wave with $\vec{E}$ perpendicular to $\vec{B}$ and with an angular frequency $\omega = 10^9$ rad/s. The confining magnetic field is 1 T (10^4 G). Is the motion of (a) the electrons and (b) the ions in response to this wave adiabatic?

9. **Energy exchange & collisions:** Consider a collision in 1-D between a massive particle M at rest and a lighter particle m with velocity v. Show that while there is significant momentum exchange in a single collision, several (of order $[M/m]$) collisions are required for energy exchange. Due to this reason the electrons and protons in a plasma can attain isotropic Maxwellian distributions relatively quickly, but electron-proton temperatures can still be different.

10. **Earth's magnetic mirror:** The Earth's magnetic dipole field is given by

$$\vec{B} = B_0 \left(\frac{r_0}{r}\right)^3 (3\cos\theta\hat{r} - \hat{z}),$$

where θ is the angle between $\hat{z}$ (the magnetic dipole direction) and $\hat{r}$. Calculate the mirror ratio $Rm \equiv B_{\rm max}/B_{\rm min} = 1/\sin^2\theta_c$ (θ_c is the critical angle of the loss cone measured at $z = 0$) as a function of the magnetic latitude on Earth's surface. Which latitudes are most effective in trapping particles?

11. **Adiabatic invariant:** Show that for a simple harmonic oscillator of a slowly varying (compared to $2\pi/\omega$) frequency ω (and therefore energy E), the orbit averaged value of E/ω is a constant.