Fluids & Plasmas, AA 363

Problem Set VII: plasmas as fluids, MHD, instabilities

1. **Electric field and hydrostatic equilibrium:** Consider an isothermal atmosphere of plasma in a constant gravitational field $\vec{g}$. Assuming the ions to be singly ionized, write down the force balance equations for the electron and ion fluids. Show that they can be combined to give the usual hydrostatic equation, but an electric field

$$\vec{E} = -\frac{m_i}{2e}\vec{g}$$

has to exist in the atmosphere to prevent charge separation. Contrast this with the case of a gas composed of lighter and heavier molecules, in which the scale height of the lighter molecules is larger. The strong electric field in a plasma created if there is charge separation ensures that the two components (electrons and ions) have the same number density. (adapted from Arnab's book)

2. **Energy conservation in MHD:** Assuming ideal MHD (i.e., no conduction, viscosity and resistivity) show that the fluid energy conservation equation can be written as

$$\frac{\partial}{\partial t}\left(\rho\epsilon + \frac{1}{2}\rho v^2\right) = -\nabla \cdot \left[\rho\vec{v}\left(w + \frac{1}{2}v^2\right)\right] - \vec{v}\cdot(\nabla \cdot \overleftrightarrow{M}),$$

where $\epsilon = p/[\rho(\gamma-1)]$ is the thermal energy per unit mass, $w = \epsilon + p/\rho$ is the enthalpy per unit mass, and $\overleftrightarrow{M} = B^2\overleftrightarrow{I}/8\pi - \vec{B}\vec{B}/4\pi$ is the Maxwell stress tensor. Obtain the equation governing the magnetic energy density ($B^2/8\pi$) using the induction equation and show that

$$\frac{\partial}{\partial t}\left(\rho\epsilon + \frac{\rho v^2}{2} + \frac{B^2}{8\pi}\right) = -\nabla \cdot \left[\vec{v}\left(\rho w + \frac{\rho v^2}{2} + \frac{B^2}{8\pi} + \vec{v}\cdot\overleftrightarrow{M}\right)\right].$$

This shows that the total energy in the box, which includes the magnetic energy, is conserved if there are no energy fluxes from the boundaries. This is true even in the presence of viscosity and resistivity (and thermal conduction) because these processes just convert kinetic and magnetic energy into thermal energy (redistribute thermal energy). (adapted from Arnab's book)

3. **Flux freezing:** Consider a constant initial magnetic field $\vec{B} = B_0\hat{y}$ in a conducting plasma. Suppose a velocity field

$$\vec{v} = v_0 e^{-y^2}\hat{x}$$

is switched on at $t = 0$. Find out how the field lines evolve at later times. Also make the sketches for filed lines at some time after switching on the velocity field. (from Arnab's book)

4. **MHD shocks:** Consider a shock wave with uniform magnetic fields B_1 and B_2 parallel to the shock front on the two sides (velocity is perpendicular to the shock front). Write down the modified Rankine-Hugoniot jump conditions (including the components of magnetic field) relating the upstream and downstream quantities.

5. **Rotating magnetized solar wind:** Consider the magnetized solar wind in the equatorial plane. Neglect viscosity and resistivity, and assume a steady state. From induction equation, show that the components of magnetic and velocity fields at a distance r are related by

$$B_\phi = \frac{v_\phi - r\Omega_\odot}{v_r}B_r,$$

where $\Omega_\odot$ is the angular velocity of the sun. From the ϕ component of the equation of motion, show further that

$$rv_\phi - \frac{B_r}{4\pi\rho v_r} rB_\phi = L$$

is a constant and can be interpreted as angular momentum per unit mass carried by the plasma and fields (remember e.m. fields carry angular momentum). Combine the two equations to show that

$$v_\phi = \Omega_\odot r \frac{M_A^2 L/r^2\Omega_\odot - 1}{M_A^2 - 1}$$

where $M_A = v_r/(\sqrt{B_r^2/4\pi\rho})$ is the Alfvén Mach number. Argue on the basis of this equation that angular momentum per unit mass carried away by the solar wind is $L = \Omega_\odot r_A^2$, where r_A is the Alfvén radius where $M_A = 1$. (from Arnab's book)

6. **Magnetorotational instability (MRI):** is a linear instability that is widely believed to be responsible for (MHD) turbulence in accretion flows (i.e., the physical cause behind the phenomenological α-viscosity). Write down the linearized MHD equations in which the mean flow is in the $\hat{\phi}$ direction given by $v_{\phi 0}(R) = R\Omega(R)$ (ignore the vertical variations in all background quantities; this is justified for modes with wavelengths much smaller than the disk scale height). Assume a background magnetic field in the $\hat{z}$ direction, $\vec{B}_0 = B_0\hat{z}$. Assume the perturbed quantities to vary as $e^{-iwt+ikz}$ (i.e., only vertical wavenumbers). Assume incompressibility $\vec{k}\cdot\vec{v}_1 = 0$, which implies that $v_{z1} = 0$ (and also that the perturbed pressure and B_{z1} vanish). Derive the following linearized equations:

$$\begin{aligned}
&-iwv_{R1} - 2\Omega v_{\phi 1} - ikv_A v_{AR1} = 0,\\
&-iwv_{\phi 1} + \frac{\kappa^2}{2\Omega} v_{R1} - ikv_A v_{A\phi 1} = 0,\\
&-iwB_{R1} - ikB_0 v_{R1} = 0,\\
&-iwB_{\phi 1} - \frac{d\Omega}{d\ln R} B_{R1} - ikB_0 v_{\phi 1} = 0,
\end{aligned}$$

where $v_A = B_0/\sqrt{4\pi\rho_0}$ ($\vec{v}_{A1} = \vec{B}_1/\sqrt{4\pi\rho_0}$) is the Alfvén (perturbed) speed, $\Omega(R)$ is the local rotation frequency and $\kappa^2 \equiv 4\Omega^2 + d\Omega^2/d\ln R$ is the local epicycle frequency. Derive the following dispersion relation for the local modes:

$$w^4 - w^2(\kappa^2 + 2k^2v_A^2) + k^2v_A^2\left(k^2v_A^2 + \frac{d\Omega^2}{d\ln R}\right) = 0.$$

Show that there are growing modes (MRI) if $d\Omega^2/dR < 0$. Show that the fastest growth rate is $|d\Omega/2d\ln R|$ and it occurs at

$$k^2v_A^2 = -\left(\frac{1}{4} + \frac{\kappa^2}{16\Omega^2}\right)\frac{d\Omega^2}{d\ln R}.$$

Calculate the fastest growth rate and the most unstable wavenumber for a Keplerian flow.