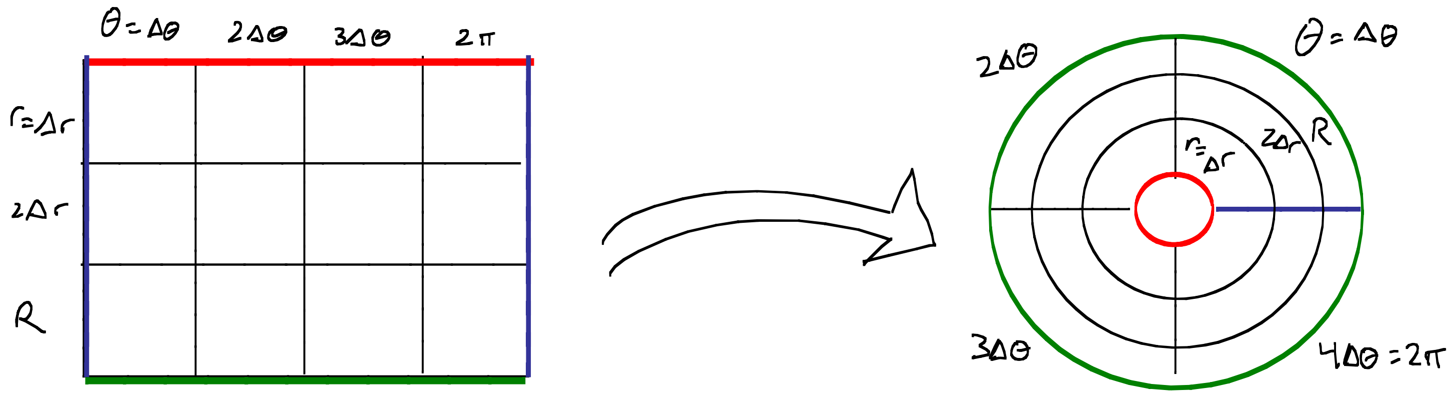
Periodic boundary conditions and false boundaries



Like finite differences in cartesian x,y coords, here we discretize $0<r<R$ and $0<\theta<2\pi$. However, constructing the matrix L^2 in exactly the same way will create unphysical boundaries in polar coordinates. We need to connect the positions w/ $\theta=\Delta\theta$ and $\theta=2\pi$ as nearest neighbors, and remove the false boundary condition close to the origin:

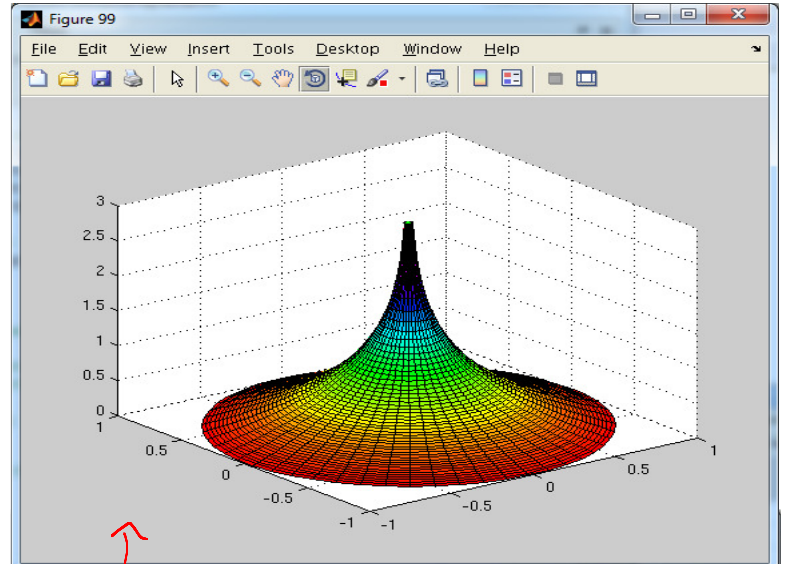
$$\frac{d^2}{d\theta^2} \longrightarrow \frac{1}{\Delta\theta^2}\begin{bmatrix} -2 & 1 & 0 & 1 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 1 & 0 & 1 & -2 \end{bmatrix}, \quad \frac{d^2}{dr^2} \rightarrow \frac{1}{\Delta r^2}\begin{array}{c} \begin{array}{ccccc} r_1 & r_2 & r_3 & \cdots & \end{array} \\ \begin{bmatrix} -1 & 1 & 0 & 0 & \cdots \\ 1 & -2 & 1 & 0 & \cdots \\ 0 & 1 & -2 & 1 & \cdots \\ & & & & \end{bmatrix} \end{array}, \quad \frac{d}{dr} \rightarrow \frac{1}{2\Delta r}\begin{array}{c} \begin{array}{ccccc} r_1 & r_2 & r_3 & \cdots & \end{array} \\ \begin{bmatrix} -2 & 2 & 0 & 0 & \cdots \\ -1 & 0 & 1 & 0 & \cdots \\ 0 & -1 & 0 & 1 & \cdots \\ & & & & \end{bmatrix} \end{array}$$

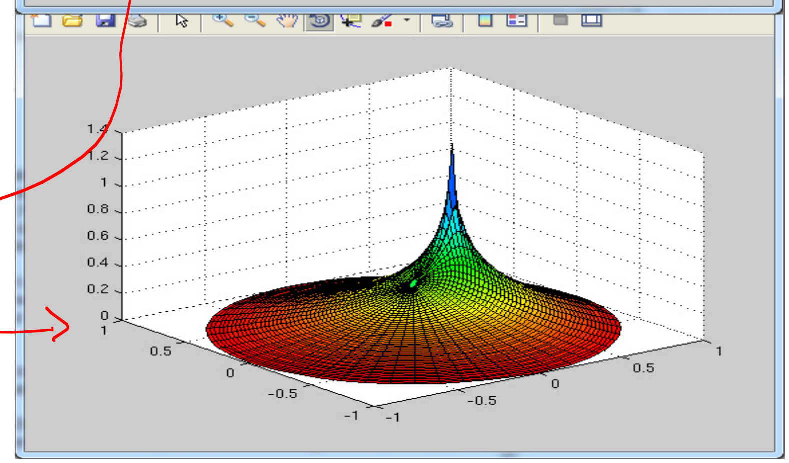
Calculation of 2D Laplace/Poisson solution in polar coords r, θ

```
clear

R=1; %radius
Nr=50; %number of radial values
Nth=51; %number of theta values

dr=R/Nr;%radial spacing
rs=linspace(dr,R,Nr); %array of radial coordinate values
r=diag(rs);%radial coordinate as a matrix operator
D1r=1/(2*dr)*(spdiags(ones(Nr,1),1,sparse(Nr,Nr))-...
    spdiags(ones(Nr,1),-1,sparse(Nr,Nr)));%1st deriv wrt radial coord r
D2r=1/dr^2*(spdiags(-2*ones(Nr,1),0,sparse(Nr,Nr))+...
    spdiags(ones(Nr,1),1,sparse(Nr,Nr))+...
    spdiags(ones(Nr,1),-1,sparse(Nr,Nr)));%2nd deriv wrt radial coord r
D1r(1,1:2)=1/dr*[-1 1];D2r(1,1)=-1/dr^2;%removes false BC at origin
Lr=D2r+r\D1r;%radial part of Laplacian in polar coords

dth=2*pi/Nth;%angular spacing
ths=linspace(dth,2*pi,Nth); %array of angular coordinates
D2th=1/dth^2*(spdiags(-2*ones(Nth,1),0,sparse(Nth,Nth))+...
    spdiags(ones(Nth,1),1,sparse(Nth,Nth))+...
    spdiags(ones(Nth,1),-1,sparse(Nth,Nth)));%2nd deriv wrt angular coord theta
D2th(1,Nth)=1/dth^2; D2th(Nth,1)=1/dth^2; %periodic boundary conditions along theta (0=2*pi)

%total Laplacian in 2D polar coords r and theta:
L2=kron(speye(Nth),r^2)\kron(D2th,speye(Nr))+kron(speye(Nth),Lr);
```

$$\frac{1}{r^2}\frac{d^2}{d\theta^2}+\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}$$

```
b=zeros(Nr,Nth);
%b(1,:)=1/dr^2; %center peak
b(round(Nr/4),1)=1/dr^2; %offset peak

A=L2\reshape(-b,Nr*Nth,1); %calculate!
Am=reshape(A,Nr,Nth); %convert solution back into a 2D array

Am=[Am(:,end) Am]; %fill in missing slice from theta=0 to dth
ths=[0 ths]; %add theta=0 to array

%convert polar coords to cartesian for plotting
[TH,R]=meshgrid(ths,rs);
[X,Y]=pol2cart(TH,R);

surf(X,Y,Am);%plot surface
```






```
%fill in surface around origin with proper color
Xp=dr*cos(ths);Yp=dr*sin(ths);
crow=round((mean(Am(1,:))-min(caxis))/diff(caxis)*length
(colormap));
cm=colormap;
patch(Xp,Yp,Am(1,:),cm(crow))
```