Unit 1 – Basic Concepts & Properties

Thomas Eiter

KBS Group, Institute of Information Systems, TU Vienna

Reasoning Web Summer School 2009

Unit Outline

Introduction

- Answer Set Programming (ASP) is a recent problem solving approach
- The term was coined by Vladimir Lifschitz [1999,2002]
- Proposed by other people at about the same time, e.g. [Marek and Truszczyński, 1999],[Niemelä, 1999]
- It has roots in KR, logic programming, and nonmonotonic reasoning
- At an abstract level, relates to SAT solving and CSP.
- Book: [Baral, 2003]

French Phrases, Italian Soda (*Dell Logic Puzzles*)

- Six people sit at a round table
- Each drinks a different kind of soda
- Each plans to visit a different French-speaking country
- The person who is planning a trip to Quebec, who drank either blueberry or lemon soda, didn't sit in seat number one.
- Jeanne didn't sit next to the person who enjoyed the kiwi soda.
- The person who has a plane ticket to Belgium, who sat in seat four or seat five, didn't order the tangelo soda.
- ...

Question:

What is each of them drinking, and where is each of them going ?

Sudoku

	6		1		4		5	
		8	3		5	6		
2								1
8			4		7			6
		6				3		
7			9		1			4
5								2
		7	2		6	9		
	4		5		8		7	

Task:

Fill in the grid so that every row, every column, and every 3x3 box contains the digits 1 through 9.

Wanted!

A general-purpose approach for modeling and solving these and many other problems.

Issues:

- Diverse domains
- Spatial and temporal reasoning
- Constraints
- Incomplete information
- Preferences and priority

Proposal:

Answer Set Programming (ASP) paradigm!

Roots of ASP – Knowledge Representation (KR)

How to model

- An agent's belief sets
- Commonsense reasoning
- Defeasible inferences
- Preferences and priority

Approach

- use a logic-based formalism
- Inherent feature: nonmonotonicity

Many logic-based KR formalisms have been developed.

Logic Programming – Prolog revisited

1960s/70s: Logic as a Programming Language (?)

- Breakthrough in Computational Logic by Robinson's discovery of the Resolution Principle (1965)

Kowalski (1979):

ALGORITHM = LOGIC + CONTROL

- Knowledge for problem solving (LOGIC)
- "Processing" of the knowledge (CONTROL)

Prolog

Prolog = "Programming in Logic"

- Basic data structures: terms
- Programs: rules and facts
- Computing: Queries (goals)
  - Proofs provide answers
  - SLD-resolution
  - unification - basic mechanism to manipulate data structures
- Extensive use of recursion

Example (Recursion)

```
append([],X,X) .
append([X|Y],Z,[X|T]) :- append(Y,Z,T) .

reverse([],[]).
reverse([X|Y],Z) :- append(U,[X],Z), reverse(Y,U) .
```

- Both relations are defined recursively.
- Terms represent complex objects: lists, sets, ...

Problem:

Reverse the list `[a,b,c]`

Ask query: `?- reverse([a,b,c],X).`

- A proof of the query yields a substitution: `X=[c,b,a]`
- The substitution constitutes an answer

The key: Techniques to search for proofs

- Understanding of the resolution mechanism is important
- It may make a difference which logically equivalent form is used (e.g., termination).

Example

```
reverse([X|Y],Z) :- append(U,[X],Z), reverse(Y,U) .
```

vs

```
reverse([X|Y],Z) :- reverse(Y,U), append(U,[X],Z) .
```

Query: `?- reverse([a|X],[b,c,d,b])`

Is this truly declarative programming?

Desiderata

Relieve the programmer from several concerns.

It is desirable that

- the order of program rules does not matter;
- the order of subgoals in a rule does not matter;
- termination is not subject to such order.

“Pure” declarative programming

- Prolog does not satisfy these desiderata
- Satisfied e.g. by the answer set semantics of logic programs

Positive Logic Programs

Definition (Positive Logic Program)

A *positive logic program* P is a finite set of clauses (rules) in the form

$$a \leftarrow b_1, \ldots, b_m \quad , \tag{1}$$

where a, b_1, ..., b_m are atoms of a first-order language L.

- a is the *head* of the rule
- $b_1, \ldots, b_m$ is the *body* of the rule.
- If $m = 0$, the rule is a *fact* (written shortly a)

Roughly, (1) can be seen as material implication $b_1 \wedge \cdots \wedge b_m \supset a$.

Example

$$connected(cagliari) \leftarrow hub(rome), link(rome, cagliari)$$
$$connected(X) \leftarrow hub(Y), link(Y, X)$$

Herbrand Semantics

Definition (Herbrand Universe, Base, Interpretation)

Given a logic program P, the **Herbrand universe** of P, $HU(P)$, is the set of all terms which can be formed from constants and functions symbols in P (resp. the vocabulary, if explicitly known).

The **Herbrand base** of P, $HB(P)$, is the set of all ground atoms which can be formed from predicates and terms $t \in HU(P)$.

A (Herbrand) **interpretation** is a first-order interpretation $I = (D, \cdot^I)$ of the vocabulary with domain $D = HU(P)$ where each term $t \in HU(P)$ is interpreted by itself, i.e., $t^I = t$.

I is identified with the set $\{ p(t_1, \ldots, t_n) \in HB(P) \mid \langle t_1^I, \ldots, t_n^I \rangle \in p^I \}$.

Informally, a (Herbrand) interpretation can be seen as a set denoting which ground atoms are true in a given scenario.

Example (Program P_1)

$$
\begin{aligned}
h(0,0).\\
t(a,b,r).\\
p(0,0,b).\\
p(f(X),Y,Z) \leftarrow p(X,Y,Z'), h(X,Y), t(Z,Z',r).\\
h(f(X),f(Y)) \leftarrow p(X,Y,Z'), h(X,Y), t(Z,Z',r).
\end{aligned}
$$

Constant symbols: 0, a, b, r; Function symbols: f.

$HU(P_1)$: { 0, a, b, r, $f(0), f(f(0))$, ... $f^i(0))$, ... $f(a), f(f(a))$, ... }

$HB(P_1)$: $\{p(0,0,0)$, $p(a,a,a)$, ... $h(0,0,)$, $h(0,a)$, ... $t(0,0,0)$, $t(a,a,a)$ }

Some Herbrand interpretations:

$$I_1 = \emptyset; \quad I_2 = HB(P_1); \quad I_3 = \{h(0,0), t(a,b,r), p(0,0,b)\}; \ldots$$

Grounding

The semantics of positive logic programs is defined in terms of grounding.

Definition (ground instance, grounding)

A *ground instance* of a clause C of the form (1) is any clause C' obtained from C by applying a substitution

$$\theta\colon Var(C) \rightarrow HU(P)$$

to the variables in C, denoted as $Var(C)$.

- $grnd(C)$ denotes the set of all possible ground instances of C
- for any program P, the *grounding of* P is $grnd(P) = \bigcup_{C \in P} grnd(C)$.

Roughly speaking, C is a shortcut denoting $grnd(C)$, and each variable appearing in C ranges over the Herbrand universe.

Example (Program P_2)

$$p(f(X), Y, Z) \leftarrow p(X, Y, Z'), h(X, Y), t(Z, Z', r).$$
$$h(0, 0).$$

The ground instances of the first rule are

$$p(f(0), 0, 0) \leftarrow p(0, 0, 0), h(0, 0), t(0, 0, r). \qquad X = Y = Z = Z' = 0$$
$$\ldots$$
$$p(f(0), r, 0) \leftarrow p(0, r, 0), h(0, r), t(0, 0, r). \qquad X = Z = Z' = 0, Y = r$$
$$\ldots$$
$$p(f(r), r, r) \leftarrow p(r, r, r), h(r, r), t(r, r, r). \qquad X = Y = Z = Z' = r$$
$$\ldots$$
$$p(f(f(0)), 0, 0) \leftarrow p(f(0), 0, 0), h(f(0), 0), t(0, 0, r). \qquad X = Y = Z = Z' = 0$$
$$\ldots$$

The single ground instance of the second rule is

$$h(0, 0).$$

Herbrand Models

Definition (Model, satisfaction)

An interpretation I is a (Herbrand) *model* of a

- a ground (variable-free) clause $C = a \leftarrow b_1, \ldots, b_m$, if either $\{b_1, \ldots, b_m\} \not\subseteq I$ or $a \in I$; $(I \models C)$
- a clause C, if $I \models C'$ for every $C' \in grnd(C)$; $(I \models C)$
- a program P, if $I \models C$ for every clause C in P. $(I \models C)$

Example (Program P_2 cont'd)

$$p(f(X), Y, Z) \leftarrow p(X, Y, Z'), h(X, Y), t(Z, Z', r).$$
$$h(0, 0).$$

Which of the following interpretations are models of P_2?

- $I_1 = \emptyset$ **no**
- $I_2 = HB(P_2)$ **yes**
- $I_3 = \{h(0,0), t(a,b,r), p(0,0,b)\}$ **no**

Which of the above interpretations are models of P_1?

Note:

Proposition

For every positive logic program P, $HB(P)$ is a model of P.

Minimal Model Semantics

- A logic program has multiple models in general.
- Select one of these models as the canonical model.
- Commonly accepted: truth of an atom in model I should be "founded" by clauses.

Example

Given

$$P_3 = \{a \leftarrow b. \quad b \leftarrow c. \quad c\},$$

truth of a in the model $I = \{a, b, c\}$ is "founded."

Given

$$P_4 = \{a \leftarrow b. \quad b \leftarrow a. \quad c\},$$

truth of a in the model $I = \{a, b, c\}$ is not founded.

Minimal Model Semantics (cont'd)

Semantics: Prefer models with true-part as small as possible.

Definition

A model I of P is *minimal*, if there exists no model J of P such that $J \subset I$.

Theorem

Every logic program P has a single minimal model (called the least *model), denoted $LM(P)$.*

This is entailed by the following property:

Proposition (Intersection closure)

If I and J are models of P, then also $I \cap J$ is a model of P.

Example

- For $P_3 = \{\ a \leftarrow b. \quad b \leftarrow c. \quad c\ \}$, we have $LM(P_3) = \{a, b, c\}$.
- For $P_4 = \{\ a \leftarrow b. \quad b \leftarrow a. \quad c\ \}$, we have $LM(P_4) = \{c\}$.
- For $P_2 = \{\ p(f(X), Y, Z) \leftarrow p(X, Y, Z'), h(X, Y), t(Z, Z', r). \quad h(0, 0)\}$, we have $LM(P_2) = \{h(0, 0)\ \}$.
- For P_1 above, we have

$$LM(P_1) = \{h(0, 0), t(a, b, r), p(0, 0, b), p(f(0), 0, a), h(f(0), f(0))\}.$$

Computation

The minimal model can be computed via fixpoint iteration.

Definition (T_P Operator)

Let $T_P\colon 2^{HB(P)} \rightarrow 2^{HB(P)}$ be defined as

$$T_P(I) = \left\{ a \;\middle|\; \begin{array}{l} \text{there exists some } a \leftarrow b_1, \ldots, b_m \\ \text{in } grnd(P) \text{ such that } \{b_1, \ldots, b_m\} \subseteq I \end{array} \right\}.$$

We let denote $\qquad T_P^0 = \emptyset, \quad T_P^{i+1} = T_P(T_P^i), \quad i \geq 0.$

Fundamental result:

Theorem

T_P has a least fixpoint, $lfp(T_P)$, and the sequence $\langle T_P^i \rangle, \quad i \geq 0$, converges to $lfp(T_P)$.

Proof: Use the fixpoint theorems of Knaster-Tarski and Kleene.

Example

- For $P_3 = \{\ a \leftarrow b. \quad b \leftarrow c. \quad c\ \}$, we have

$$T^0_{P_3} = \{\},\ T^1_{P_3} = \{c\},\ T^2_{P_3} = \{c, b\},\ T^3_{P_3} = \{c, b, a\},\ T^4_{P_3} = T^3_{P_3}$$

  Hence $\mathit{lfp}(T_{P_3}) = \{c, b, a\}$

- For $P_4 = \{\ a \leftarrow b. \quad b \leftarrow a. \quad c\ \}$, we have

$$T^0_{P_4} = \{\}, \quad T^1_{P_4} = \{c\}, \quad T^2_{P_4} = T^1_{P_4}$$

  Hence $\mathit{lfp}(T_{P_4}) = \{c\}$

- For program P_2 above, we have

$$T^0_{P_2} = \emptyset, \quad T^1_{P_2} = \{h(0,0)\}, \quad T^2_{P_2} = T^1_{P_2}.$$

  Hence $\mathit{lfp}(T_{P_2}) = \{h(0,0)\}$.

Example (cont'd)

- For program P_1 above, we have

$$\begin{aligned} T^0_{P_1} &= \emptyset, \\ T^1_{P_1} &= \{h(0,0), t(a,b,r), p(0,0,b)\}, \\ T^2_{P_1} &= \{h(0,0), t(a,b,r), p(0,0,b), p(f(0),0,b), h(f(0),f(0))\}, \\ T^2_{P_1} &= T^3_{P_1}. \end{aligned}$$

Hence
$lfp(T_{P_1}) = \{h(0,0), t(a,b,r), p(0,0,b), p(f(0),0,b), h(f(0),f(0))\}$.

- For program $P = \{p(0). \quad p(f(X)) \leftarrow p(X)\}$, we have

$$T^0_P = \emptyset, \quad T^1_P = \{p(0)\}, \ldots, T^i_P = \{p(0), \ldots p(f^{i-1}(0))\}, \quad i \geq 0;$$

hence $lfp(T_P) = \{p(f^i(0)) \mid i \geq 0\}$ is infinite.

Negation in Logic Programs

Why negation?

- Natural linguistic concept
- Facilitates convenient, declarative descriptions (definitions)

  E.g., "Men who are not husbands are singles.”

Definition

A *normal logic program* is a set of rules of the form

$$a \leftarrow b_1, \ldots, b_m, not\, c_1, \ldots, not\, c_n \qquad (n, m \geq 0) \tag{2}$$

where a and all b_i, c_j are atoms in a first-order language L.

not is called “negation as failure”, “default negation”, or “weak negation”

Things get more complex!

Programs with Negation

Prolog: "$not\ \langle X\rangle$" means "Negation as Failure (to prove to $\langle X\rangle$)"

Different from negation in classical logic!

Example (Program P_5)

$$man(dilbert).$$
$$single(X) \leftarrow man(X), not\ husband(X).$$
$$husband(X) \leftarrow fail.$$ % fail = "false" in Prolog

Query:

$$?-single(X).$$

Answer:

$$X = dilbert\ .$$

Example (cont'd)

Modifying the last rule of P_5, we get P_6:

$$\begin{aligned} man(dilbert). \\ single(X) \leftarrow man(X), not\, husband(X). \\ husband(X) \leftarrow man(X), not\, single(X). \end{aligned}$$

Result in Prolog ????

Problem: not a single intuitive model!

Two intuitive Herbrand models:

$$M_1 = \{man(dilbert), single(dilbert)\}, \text{ and}$$
$$M_2 = \{man(dilbert), husband(dilbert)\} \ .$$

Which one to choose?

Semantics of Logic Programs With Negation

- "War of Semantics" in Logic Programming (1980/90ies):

  Meaning of programs like the Dilbert example above

- Great Schism: Single model vs. multiple model semantics

- To date:
  - *Well-Founded Semantics* [Van Gelder *et al.*, 1991]

    Partial model: $man(dilbert)$ is true,
    $single(dilbert)$, $husband(dilbert)$ are unknown

  - *Answer Set (alias Stable Model) Semantics* by Gelfond and Lifschitz [1988,1991].

    Alternative models: $M_1 = \{man(dilbert), single(dilbert)\}$,
    $M_2 = \{man(dilbert), husband(dilbert)\}$.

- Agreement for so-called "stratified programs"

  Different selection principles for non-stratified programs

Stratified Negation

Intuition: To evaluate a rule r: $a \leftarrow \cdots, not\, p(\vec{t}), \cdots$, the value of $p(\vec{t})$ should be known.

1. Evaluate first $p(\vec{t})$.
2. if $p(\vec{t})$ is $\begin{cases} \text{false,} & \text{then } not\, p(\vec{t}) \text{ is true,} \\ \text{true,} & \text{then } not\, p(\vec{t}) \text{ is false and } r \text{ is not applicable.} \end{cases}$

Example

$$P = \{\ boring(chess) \leftarrow not\, interesting(chess)\ \}$$

- $interesting(chess)$ is false $\Rightarrow$ $not\, interesting(chess)$ is true.
- hence, r is applied and $boring(chess)$ is true.
- This leads to the Herbrand model $H = \{boring(chess)\}$ of P.

Note: this introduces *procedurality* (violates declarativity)!

Dependency Graph

Restriction:

- The method works if there is no cyclic negation.
- Need a syntactic criterion to ensure this property.

Definition (Dependency graph)

The *dependency graph* of a set P of rules, is a directed graph $dep(P) = \langle N, E \rangle$ where

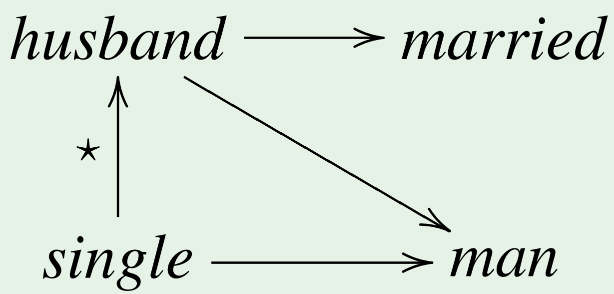
- N = { predicate $p \mid p$ occurs in P, p is not a built-in } (=: $pred(S)$)
- E contains $p \rightarrow q$, iff P contains some rule $a \leftarrow \cdots, \ell, \cdots$ with $a = p(\cdots)$ and $\ell = q(\cdots)$ or $\ell = not\, q(\cdots)$

  Label $p \rightarrow q$ with "*", if $\ell = not\, q(\cdots)$

Example (Program P_7)

$$man(dilbert).$$
$$husband(X) \leftarrow man(X),\ married(X).$$
$$single(X) \leftarrow man(X), not\, husband(X).$$

$dep(P_7)$:



Stratification

Definition (Stratification)

A *stratification* of a set P of rules is a partitioning

$$\Sigma = \{S_1 \dots, S_n\}$$

of $pred(P)$ into n nonempty, pairwise disjoint sets such that

(a) if $p \in S_i$, $q \in S_j$, and $p \rightarrow q$ is in $dep(P)$, then $i \geq j$; and

(b) if $p \in S_i$, $q \in S_j$, and $p \rightarrow^{\star} q$ is in $dep(P)$ then $i > j$.

The sets $S_1, \dots, S_n$ are the *strata* of P w.r.t. Σ.

P is *stratified*, if it has some stratification Σ.

- Informally, Σ specifies an *order of evaluation* for the predicates in P
- The sequential evaluation of S_1, S_2,...,S_n can be done by computing a series of *iterative least models*.

Semantics

Definition (iterative least model)

Suppose P is a logic program with stratification $\Sigma = \{S_1, \ldots, S_k\}$, $k \geq 1$. Then

- $P_{S_i} = \{a \leftarrow b_1, \ldots, b_n \in P \mid a = p(\cdots), p \in S_i\}$, and
- $HB^\star(P_{S_i}) = \bigcup_{j \leq i}\{p(\mathbf{t}) \in HB(P) \mid p \in S_j\}$.

The *iterative least models* $M_i \subseteq HB(P)$, $1 \leq i \leq k$, are such that

(i) M_1 is the least model of P_{S_1};

(ii) if $i > 1$, then M_i is the least subset M of $HB(P)$ such that
- M is a model of P_{S_i}, and
- $M \cap HB^\star(P_{S_{i-1}}) = M_{i-1} \cap HB^\star(P_{S_{i-1}})$.

The *iterative least model* of P is $M_{P,\Sigma} = M_k$.

Example (P_7 cont'd)

$$
\begin{aligned}
man(dilbert).\\
husband(X) \leftarrow man(X),\ married(X).\\
single(X) \leftarrow man(X),\ not\, husband(X).
\end{aligned}
$$

Stratification: $\Sigma = \{S_1 = \{man, married\},\ S_2 = \{husband\},\ S_3 = \{single\}\}$

- $P_{S_1} = \{man(dilbert)\}$ and $M_1 = LM(P_{S_1}) = \{man(dilbert)\}$.
- $P_{S_2} = \{husband(X) \leftarrow man(X), married(X)\}$.
  $HB^\star(P_{S_1}) = \{man(dilbert), married(dilbert)\}$.
  Then, $M_2 = \{man(dilbert)\}$ is a model of P_{S_2} and
  $M_2 \cap HB^\star(P_{S_1}) = M_1 \cap HB^\star(P_{S_1})$ (no smaller such model exists)
- $P_{S_3} = \{single(X) \leftarrow man(X),\ not\, husband(X)\}$.
  Thus $M_3 = \{single(dilbert)\} \cup M_2$ is the least model of P_{S_3} such that
  $M_3 \cap HB^\star(P_{S_2}) = M_2 \cap HB^\star(P_{S_2})$.

Stratification Theorem

Note: stratifications are not unique.

Example (P_7 cont'd)

Other stratification: $\Sigma' = \{S'_1 = \{man, married, husband\},\ S'_2 = \{single\}\}$.

Evaluation with Σ' yields same result!

This is not accidental:

Theorem (Apt *et al.* [1988])

Let P be a stratified program. Then for every stratifications Σ and Σ' of P, it holds that $M_{P,\Sigma} = M_{P,\Sigma'}$.

Hence, we can simplify $M_{P,\Sigma}$ to $M_P = M_{P,\Sigma}$ (for arbitrary Σ of choice)

Corollary

Stratified programs have a canonical model, also called perfect model.

Stable model semantics

First, for variable-free (ground) programs P

- Treat "*not*" specially
- Intuitively, literals *not a* are a source of "contradiction" or "unstability".

Example (P_6 cont'd)

$$man(dilbert). \qquad (f_1)$$

$$single(dilbert) \leftarrow man(dilbert), not\ husband(dilbert). \qquad (r_1)$$

$$husband(dilbert) \leftarrow man(dilbert), not\ single(dilbert). \qquad (r_2)$$

- Consider $M' = \{man(dilbert)\}$.

  If as in M', $man(dilbert)$ were true and $husband(dilbert)$ false, by r_1 also $single(dilbert)$ should be true. This is not coherent.

- Consider $M'' = \{man(dilbert), single(dilbert), husband(dilbert)\}$.

  The bodies of r_2 and R_2 are not true wrt M'', hence there is no evidence for $single(dilbert)$ and $husband(dilbert)$ being true.

Stable Models

Definition (Gelfond-Lifschitz Reduct P^M 1988)

The *GL-reduct* (simply *reduct*) of a ground program P w.r.t. an interpretation M, denoted P^M, is the program obtained from P by

1. removing rules with $not\, a$ in the body for each $a \in M$; and
2. removing literals $not\, a$ from all other rules.

Intuition:

- M makes an **assumption** about what is true and what is false.
- The reduct P^M incorporates this assumptions.
- As a "not"-free program, P^M derives positive facts, given by $LM(P^M)$.
- If this coincides with M, then the assumption of M is "stable".

Stable Models (cont'd)

Definition (stable model)

An interpretation M of P is a *stable* model of P, if

$$M = LM(P^M).$$

Observe:

- $P^M = P$ for any "not"-free program P.
- Thus, for any positive program $LM(P)$ (=$LM(P^M)$) is its single stable model.

Example (P_6 cont'd)

$$man(dilbert). \qquad (f_1)$$

$$single(dilbert) \leftarrow man(dilbert), not\ husband(dilbert). \qquad (r_1)$$

$$husband(dilbert) \leftarrow man(dilbert), not\ single(dilbert). \qquad (r_2)$$

Candidate interpretations:

- $M_1 = \{man(dilbert), single(dilbert)\}$,
- $M_2 = \{man(dilbert), husband(dilbert)\}$,
- $M_3 = \{man(dilbert), single(dilbert), husband(dilbert)\}$
- $M_4 = \{man(dilbert)\}$,

M_1 and M_2 are stable models.

Example (P_6 cont'd)

$$man(dilbert). \qquad (f_1)$$

$$single(dilbert) \leftarrow man(dilbert), not\, husband(dilbert). \qquad (r_1)$$

$$husband(dilbert) \leftarrow man(dilbert), not\, single(dilbert). \qquad (r_2)$$

- $M_1 = \{man(dilbert), single(dilbert)\}$:

  reduct $P_6^{M_1}$:

  $$man(dilbert).$$

  $$single(dilbert) \leftarrow man(dilbert).$$

  The least model of $P_6^{M_1}$ is $\{man(dilbert), single(dilbert)\} = M_1$.

- $M_2 = \{man(dilbert), husband(dilbert)\}$: by symmetry of *husband* and *single*, also M_2 is stable.

Example (P_6 cont'd)

$$man(dilbert). \quad (f_1)$$
$$single(dilbert) \leftarrow man(dilbert), not\, husband(dilbert). \quad (r_1)$$
$$husband(dilbert) \leftarrow man(dilbert), not\, single(dilbert). \quad (r_2)$$

- $M_3 = \{man(dilbert), single(dilbert), husband(dilbert)\}$:

  $P_6^{M_3}$ is

  $$man(dilbert).$$

  $LM(P_6^{M_3}) = \{man(dilbert)\} \neq M_3$.

- $M_4 = \{man(dilbert)\}$:

  $P_6^{M_4}$ is

  $$man(dilbert).$$
  $$single(dilbert) \leftarrow man(dilbert).$$
  $$husband(dilbert) \leftarrow man(dilbert).$$

  $LM(P_6^{M_4}) = \{man(dilbert), single(dilbert), husband(dilbert)\} \neq M_4$.

Inconsistent Programs

- Each normal logic program has some Herbrand model.
- However, it may have no stable model.

Example ($P_\perp$)

$$p \leftarrow not\, p$$

- Candidate interpretations: $M_1 = \{\}$, $M_2 = \{p\}$.
- M_1: $P_\perp^{M_1} = \{p\}$, and $LM(P_\perp) = \{p\} \neq M_1$.
- M_2: $P_\perp^{M_2} = \{\}$, and $LM(P_\perp) = \{\} \neq M_2$.

Note:

- If p does not occur in P, then $P \cup \{p \leftarrow not\, p\}$ has no stable model.
- Adding $p \leftarrow not\, p$ to P "kills" all stable models of P !

Programs with Variables

- Consider, like in Prolog, only Herbrand interpretations.
- As for positive programs, view a program clause as a shorthand for all its ground instances.
- Recall: $grnd(P)$ is the grounding of program P.

Definition (stable model, general case)

An interpretation M of P is a *stable model* of P, if M is a stable model of $grnd(P)$.

- Alternative way: Perform grounding in the GL-reduct, i.e., require $M = LM(P^M)$ where $P^M =_{def} grnd(P)^M$ for non-ground P.

Example (Variant P'_6 of P_6)

$$man(dilbert). \quad (r_1)$$
$$woman(alice). \quad (r_2)$$
$$single(X) \leftarrow man(X), not\ husband(X). \quad (r_3)$$
$$husband(X) \leftarrow man(X), not\ single(X). \quad (r_4)$$

We have that, for instance,

$$grnd(r_3) = \{ \ single(dilbert) \leftarrow man(dilbert), not\ husband(dilbert).$$
$$single(alice) \leftarrow man(alice), not\ husband(alice). \quad \};$$

$$grnd(P'_6) = \{ \ man(dilbert).$$
$$woman(alice).$$
$$single(dilbert) \leftarrow man(dilbert), not\ husband(dilbert).$$
$$single(alice) \leftarrow man(alice), not\ husband(alice).$$
$$husband(dilbert) \leftarrow man(dilbert), not\ single(dilbert).$$
$$husband(alice) \leftarrow man(alice), not\ single(alice). \quad \}.$$

Example (P'_6 cont'd)

The program $grnd(P'_6)$, and thus P'_6, has the following stable models:

- $M_1 = \{man(dilbert), woman(alice), single(dilbert)\}$
- $M_2 = \{man(dilbert), woman(alice), husband(dilbert)\}$

Indeed,

- the rule instances of r_3 and r_4 for *dilbert* generate two possible scenarios;
- the rule instances of r_3 and r_4 for *alice* are inapplicable.

Properties of Stable Models

- Stable model semantics has a strong theoretical basis, many properties are known.
- We consider here some elementary ones.
- See e.g.
  - [Lifschitz, 2008]
  - [Ferraris and Lifschitz, 2005]
  - [Gelfond, 2008]

  for other insights, alternative definitions and properties of stable models.

Relationship to Classical Models

How do stable models of P relate to classical models of P?

Definition (classical model of normal logic program)

An interpretation I is a *model* of

- a ground clause $C : a \leftarrow b_1, \ldots, b_m, not\, c_1, \ldots, not\, c_n$, if either $\{b_1, \ldots, b_m\} \not\subseteq I$ or $\{a, c_1, \ldots, c_n\} \cap I \neq \emptyset$ $(I \models C)$;
- a clause C, if $I \models C'$ for every $C' \in grnd(C)$ $(I \models C)$;
- a set P of rules, if $I \models C$ for every clause C in P $(I \models P)$.

This complies with Herbrand models satisfying the clause

$$a \vee not\, b_1 \vee \ldots \vee not\, b_m \vee c_1 \vee \ldots \vee c_n,$$

where not is interpreted as classical negation ("$\neg$").

Relationship to Classical Models (cont'd)

The following holds:

Theorem

1. *Every stable model M of P is a model of P.*
2. *A stable model M does not contain any model M' of P properly ($M' \not\subset M$), i.e., is a minimal model of P (w.r.t. $\subseteq$).*

Corollary

Stable models are incomparable w.r.t. $\subseteq$, i.e., if M_1 and M_2 are different stable models of P, then $M_1 \not\subseteq M_2$ and $M_2 \not\subseteq M_1$.

Thus, stable models adhere to minimality of positive information.

Supportedness

- Note: each atom a in a stable model M must be derived from some rule of P.
- Extend the immediate consequence operator T_P to not .

Definition (T_P for normal P)

Given a normal program P and an interpretation I, let

$$T_P(I) = \left\{ a \;\middle|\; \begin{array}{l} \text{there is some } r = a \leftarrow b_1, \ldots, b_m, not\, c_1, \ldots, not\, c_n \in grnd(P) \\ \text{such that } \{b_1, \ldots, b_m\} \subseteq I, \{c_1, \ldots c_m\} \cap I = \emptyset \end{array} \right\} .$$

An interpretation I of P is a *supported model* of P, if $T_P(I) = I$.

Theorem

Every stable model M of P is a supported model of P.

Supportedness (cont'd)

- In fact, by minimality of stable models, every stable model is a minimal (w.r.t. $\subseteq$) supported model of P.
- The converse is not true.

Example (Program P_s)

$$a \leftarrow not\, b.$$
$$b \leftarrow c.$$
$$c \leftarrow b.$$

- Note that $M_1 = \{a\}$ and $M_2 = \{b, c\}$ are both minimal such that $T_{P_s}(M_1) = M_1$ and $T_{P_s}(M_2) = M_2$.
- The single stable model of P_s is M_1.
- Problem with M_2: Self-supportedness of b (via c)

Unfounded sets

Stable models amount to supported models with no (cyclic) self-support

Definition (cf. [Van Gelder *et al.*, 1991],[Leone *et al.*, 1997])

A set $U \subseteq HB_P$ is an *unfounded set* of P relative to interpretation I, if for every $a \in U$ and $r : a \leftarrow b_1, \ldots, b_m, not\, c_1, \ldots, not\, c_n$ in $\in grnd(P)$, either

1. for some $i \in \{1, \ldots, m\}$, either $b_i \notin I$ or $b_i \in U$, or
2. for some $j \in \{1, \ldots, n\}$, $c_j \in I$.

- Every P has a *greatest unfounded set* relative to I, denoted $U_P(I)$.
- Intuitively, if I is compatible with P, all atoms in $U_P(I)$ can be safely switched to false while maintaining compatibilty.

Definition (unfounded-freeness)

I is called *unfounded-free*, if $I \cap U = \emptyset$ for each unfounded set U of P rel. to I.

Note: I is *unfounded-free* iff $I \cap U_P(I) = \{\}$.

Unfounded sets (cont'd)

Theorem (implicit in [Leone *et al.*, 1997])

Given a program P, a model M of P is stable iff M is unfounded-free.

Example (P_s cont'd)

$$a \leftarrow not\, b.$$
$$b \leftarrow c.$$
$$c \leftarrow b.$$

- $M_2 = \{b, c\}$: $U_{P_s}(M_2) = \{b, c\}$, thus $M_2 \cap U_{P_s}(M_2) \neq \emptyset$.
- $M_1 = \{a\}$: $U_{P_s}(M_1) = \emptyset$, thus $M_1 \cap U_{P_s}(M_1) = \emptyset$.

- Unfounded-freeness is exploited for computing stable models (DLV)
- It corresponds to loop formulas [Lin and Zhao, 2002], [Lee, 2005].

Stratified Programs

Stable model semantics gracefully generalizes stratified semantics:

Theorem

If a program P is stratified, then P has a single stable model, which coincides with the perfect (i.e., the iterative least) model model of P.

Notes:

- A stratified P may have several minimal models; only one is stable

  E.g., $P = \{boring(chess) \leftarrow not\ interesting(chess)\}$

  has two minimal models:

  $$M_1 = \{boring(chess)\} \text{ and } M_2 = \{interesting(chess)\}.$$

  The perfect model is $M_P = M_1$.

- Stratified programs can only express deterministic scenarios, no "alternatives" are possible!

Non-Cumulativity

- In classical logic, adding consequences of a theory T to T preserves its semantics.
- This property is known as cumulativity (or lemma support).
- For stable model semantics, this property does not hold.

Proposition

Suppose P and atom a fulfill $M \models a$, for each stable model M of P. Then P and $P \cup \{ a \}$ need not have the same stable models (even if P is consistent).

Example

$$b \leftarrow not\, c. \quad c \leftarrow not\, b.$$

$$a \leftarrow b. \quad a \leftarrow not\, a.$$

P has the stable model $M = \{a, b\}$; $P \cup \{a\}$ has in addition $N = \{a, c\}$.

Note: the property holds for stratified programs.

Computational Properties

How difficult is it to compute some stable model?

Decision problem CONS:

Given a program P, does P have some stable model?

Theorem

For normal logic programs P*, problem* CONS *is*

- NP*-complete in the propositional and ground case;*
- NEXPTIME*-complete in the datalog (function-free) case;*
- Σ_1^1*-complete in the general first-order case.*

Recall: NP (NEXPTIME) = class of problems solvable in polynomial (exponential) time on a non-deterministic Turing machine.

Σ_1^1 is a class in the Analytic Hierarchy

Computational Properties (cont'd)

Lower complexity holds for fragments:

- For positive and stratified propositional programs, CONS is polynomial (in fact, trivial).
  - Still solvable in linear time if constraints and strong negation are allowed (P-complete).
  - For datalog programs, complexity increases to EXPTIME.
- For programs with function symbols, several decidable program classes are known (up to 3-EXPTIME).

More on basic complexity: [Dantsin *et al.*, 2001].

Extensions

- Many extensions exist, partly motivated by applications
- Some are syntactic sugar, other strictly add expressiveness
- Incomplete list:
  - constraints
  - strong negation
  - disjunction
  - nested expressions
  - cardinality constraints (Smodels)
  - optimization: weight constraints, *minimize* (Smodels); weak constraints (DLV)
  - aggregates (Smodels, DLV)
  - templates (for macros), external functions (DLVHEX)
  - Frame Logic syntax (for Semantic Web)
  - preferences: e.g., PLP
  - KR frontends (diagnosis, inheritance, planning,...) in DLV
- Comprehensive survey: [Niemelä (ed.), 2005]

Constraints

- Adding

$$p \leftarrow q_1, \ldots, q_m, not\, r_1, \ldots, not\, r_n, not\, p.$$

  to P “kills” all stable models of P that
  - contain $q_1, \ldots, q_m$, and
  - do not contain $r_1, \ldots, r_n$

- This is convenient to eliminate scenarios which does not satisfy integrity constraints.

- Short:

Constraint

$$\leftarrow q_1, \ldots, q_m, not\, r_1, \ldots, not\, r_n.$$

Example (Dilbert P_6 cont'd)

$$man(dilbert). \qquad (f_1)$$

$$single(dilbert) \leftarrow man(dilbert), not\ husband(dilbert). \qquad (r_1)$$

$$husband(dilbert) \leftarrow man(dilbert), not\ single(dilbert). \qquad (r_2)$$

$$\leftarrow husband(X), not\ wedding\_ring(X). \qquad (c_1)$$

- The constraint c_1 eliminates models in which there is no evidence for a husband having a wedding ring.
- Single stable model: $M_1 = \{man(dilbert), single(dilbert)\}$

Strong Negation

- Weak negation "*not* a" means "a can not be proved (derived) using rules," and that a is false by default (believed to be false).
- This is different from *knowing* (provably) that a is false; this is expressed by $-a$ (sometimes $\neg a$).
- This is called *strong negation* and may make an important difference.

Example (due to John McCarthy)

Consider an agent A with the following task:

- "At a railroad crossing, cross the rails if no train approaches."

We may encode this scenario using one of the following two rules:

$$walk \leftarrow at(A, L), crossing(L), not\ train\_approaches(L). \quad (r_1)$$

$$walk \leftarrow at(A, L), crossing(L), -train\_approaches(L). \quad (r_2)$$

Extended Logic Programs

Definition

An *extended logic program (ELP)* is a finite set of rules

$$a \leftarrow b_1, \ldots, b_m, not\, c_1, \ldots, not\, c_n \qquad (n, m \geq 0) \tag{3}$$

where a, b_i, c_j are atoms or strongly negated atoms in a f.o. language L.

The semantics of ELPs can be defined by program transformation:

- view literals "$-p(\vec{X})$" as atoms with fresh predicate symbols "$-p$";
- add clauses $falsity \leftarrow not\, falsity, p(\vec{X}), -p(\vec{X})$ to P ($p(\vec{X})$ and $-p(\vec{X})$ are not simultaneously true); and
- select the stable models of the resulting program (called *answer sets* of P).

Answer sets M: *three-valued view*

Atom a may be true ($a \in M$), false ($-a \in M$), or *unknown* ($a, -a \notin M$).

Strong negation (combined with weak negation) is e.g. helpful to express *default rules*.

Example (French speaking)

$$french(luc). \qquad (f_1)$$

$$speaks(X, french) \leftarrow french(X), not\ -speaks(X, french). \qquad (r_1)$$

$$-speaks(X, french) \leftarrow thumb(X). \qquad (r_2)$$

- r_1 expresses that by default, French can speak French.
- Single answer set $M = \{french(luc), speaks(luc, french)\}$.

Note:

- ELPs are closely related to Default Logic [Reiter, 1980]
- The answer sets of P correspond 1-1 to the extensions of the default theory $T = (\emptyset, \{d(C) \mid C \in P\})$ ($d(C)$ casts C into a default rule).

Disjunction

The use of disjunction is natural to express indefinite knowledge.

Example

- $female(X) \vee male(X) \leftarrow person(X).$
- $broken(left\_hand, tom) \vee broken(right\_hand, tom).$

Disjunction is natural for expressing a "guess" and to create non-determinism

Example

- $ok(C) \vee -ok(C) \leftarrow component(C).$

Minimality

- Semantics: disjunction is *minimal* (different from classical logic):

$$a \vee b \vee c.$$

  Minimal models: $\{a\}$, $\{b\}$, and $\{c\}$.

- actually *subset minimal*:

$$a \vee b. \qquad a \vee c.$$

  Minimal models: $\{a\}$ and $\{b, c\}$.

$$a \vee b. \qquad a \leftarrow b$$

  Models $\{a\}$ and $\{a, b\}$, but only $\{a\}$ is minimal.

- but minimality is *not necessarily exclusive*:

$$a \vee b. \qquad b \vee c. \qquad a \vee c.$$

  Minimal models: $\{a, b\}$, $\{a, c\}$, and $\{b, c\}$.

Disjunction vs. Unstratified Negation

Reconsider the Dilbert Program P_6:

$$\begin{aligned} man(dilbert). \\ single(X) \leftarrow man(X), not\ husband(X). \\ husband(X) \leftarrow man(X), not\ single(X). \end{aligned}$$

is under stable semantics equivalent to the program P_{dd}:

$$\begin{aligned} man(dilbert). \\ single(X) \vee husband(X) \leftarrow man(X). \end{aligned}$$

The use of disjunction is more intuitive!

Extended Logic Programs with Disjunctions

Definition

A *extended disjunctive logic program* (EDLP) is a finite set of rules

$$a_1 \vee \cdots \vee a_k \leftarrow b_1, \ldots, b_m, not\, c_1, \ldots, not\, c_n \qquad (k, m, n \geq 0) \tag{4}$$

where all a_i, b_j, c_l are atoms or strongly negated atoms in f.o. language L.

Semantics:

- Answer sets of P are defined similarly as for an ELP
- Differences:
  - I is a model of ground (4), if either $\{b_1, \ldots, b_m\} \not\subseteq I$ or $\{a_1, \ldots, a_k, c_1, \ldots, c_n\} \cap I \neq \emptyset$
  - "M is the least model of P^M" $\rightsquigarrow$ "M is a minimal model of P^M" (P^M may have multiple minimal models).

Example (Disjunctive Dilbert P_{dd}, cont'd)

$$man(dilbert).$$
$$single(X) \lor husband(X) \leftarrow man(X).$$

As P_{dd} is "*not*"-free, $grnd(P_{dd})^M = grnd(P_{dd})$ for every M.

Answer sets:

- $M_1 = \{man(dilbert), single(dilbert)\}$, and
- $M_2 = \{man(dilbert), husband(dilbert)\}$.

Some Properties of EDLPs

- Every answer set of an EDLP P is a minimal model of P (models analogous as for ELPs)
- Different answer sets of an EDLP P are incomparable
- An EDLP may have no, a single or multiple answer sets
- For EDLPs without strong negation, answer sets are models that are unfounded-free [Leone *et al.*, 1997]
- Deciding whether a propositional EDLP P has some answer set is Σ_2^p-complete. ($\Sigma_2^p = \mathrm{NP}^{\mathrm{NP}}$)

> Disjunction adds higher problem solving capacity, it is not just syntactic sugar!

- **But:** EDLPs **can not** be regarded as a fragment of Reiter's Default Logic.

References I

- K.R. Apt, H.A. Blair, and A. Walker.
  Towards a Theory of Declarative Knowledge.
  In J. Minker, editor, *Foundations of Deductive Databases and Logic Programming*, pages 89–148. Morgan Kaufman, Washington DC, 1988.

- Chitta Baral.
  Knowledge Representation, Reasoning and Declarative Problem Solving.
  Cambridge University Press, 2003.

- Evgeny Dantsin, Thomas Eiter, Georg Gottlob, and Andrei Voronkov.
  Complexity and Expressive Power of Logic Programming.
  ACM Computing Surveys, 33(3):374–425, 2001.

- Paolo Ferraris and Vladimir Lifschitz.
  Mathematical foundations of answer set programming.
  In *We Will Show Them! Essays in Honour of Dov Gabbay, Volume One*, pages 615–664. College Publications, 2005.

References II

- Michael Gelfond and Vladimir Lifschitz.

  The Stable Model Semantics for Logic Programming.

  In *Logic Programming: Proceedings Fifth Intl Conference and Symposium*, pages 1070–1080, Cambridge, Mass., 1988. MIT Press.

- Michael Gelfond and Vladimir Lifschitz.

  Classical Negation in Logic Programs and Disjunctive Databases.

  New Generation Computing, 9:365–385, 1991.

- M. Gelfond.

  Answer sets.

  In B. Porter F. van Harmelen, V. Lifschitz, editor, *Handbook of Knowledge Representation*, chapter 7, pages 285–316. Elsevier, 2008.

References III

Joohyung Lee.

A model-theoretic counterpart of loop formulas.

In Leslie Pack Kaelbling and Alessandro Saffiotti, editors, *IJCAI*, pages 503–508. Professional Book Center, 2005.

Nicola Leone, Pasquale Rullo, and Francesco Scarcello.

Disjunctive Stable Models: Unfounded Sets, Fixpoint Semantics and Computation.

Information and Computation, 135(2):69–112, June 1997.

Vladimir Lifschitz.

Answer set planning.

In *ICLP*, pages 23–37, 1999.

Vladimir Lifschitz.

Answer Set Programming and Plan Generation.

Artificial Intelligence, 138:39–54, 2002.

References IV

- Vladimir Lifschitz.

  Twelve definitions of a stable model.

  In *ICLP*, pages 37–51, 2008.

- Fangzhen Lin and Yuting Zhao.

  ASSAT: Computing Answer Sets of a Logic Program by SAT Solvers.

  In *AAAI/IAAI*, pages 112–, 2002.

- Victor W. Marek and Mirosław Truszczyński.

  Stable Models and an Alternative Logic Programming Paradigm.

  In K. Apt, V. W. Marek, M. Truszczyński, and D. S. Warren, editors, *The Logic Programming Paradigm – A 25-Year Perspective*, pages 375–398. Springer, 1999.

References V

Ilkka Niemelä (ed.).

Language Extensions and Software Engineering for ASP.

Technical Report WP3, Working Group on Answer Set Programming (WASP), IST-FET-2001-37004, September 2005.

Available at `http://www.tcs.hut.fi/Research/Logic/wasp/wp3/wasp-wp3-web/`.

Ilkka Niemelä.

Logic Programming with Stable Model Semantics as Constraint Programming Paradigm.

Annals of Mathematics and Artificial Intelligence, 25(3–4):241–273, 1999.

Raymond Reiter.

A Logic for Default Reasoning.

Artificial Intelligence, 13(1–2):81–132, 1980.

References VI

Allen Van Gelder, Kenneth A. Ross, and John S. Schlipf.
The Well-Founded Semantics for General Logic Programs.
Journal of the ACM, 38(3):620–650, 1991.